

## Physical systems

### Mechanical systems

Mechanical oscillations are of critical importance for the stability of a structure. Building structures all have specific natural frequencies and these frequencies can be influenced in various ways.

Resonant frequency (natural frequency): more and more energy is combined in a specific vibration as a function in time, leading to an increasing amplitude.

$$\#degrees = \#radians * (180^\circ/\pi)$$

Simple harmonic motions occur throughout the natural world. A simple harmonic motion follows a straight line (the x-direction).

$$x(t) = A * \sin(\omega t + \alpha)$$

A is the maximum displacement: amplitude

$\omega$  is the angular frequency,  $\omega = 2\pi f = 2\pi/T$  (f = frequency)

$\alpha$  is the phase at time  $t = 0$

$$\sin \alpha = \sin(\alpha + n2\pi)$$

The displacement is maximum in the positive x-direction when  $\sin(\omega t + \alpha) = 1$ , which happens when  $\omega t + \alpha = 1/2\pi + n2\pi$ . The displacement is then equal to the amplitude A.

The displacement is maximum in the negative x-direction when  $\sin(\omega t + \alpha) = -1$ , which happens when  $\omega t + \alpha = 3/2\pi + n2\pi$ . The displacement is then equal to the negative amplitude -A.

The speed of a particle is the derivative of the position:

$$v = \frac{dx}{dt} = \frac{d[A * \sin(\omega t + \alpha)]}{dt} = \omega A * \cos(\omega t + \alpha)$$

By maximum displacement,  $x = A$  and therefore  $\omega t + \alpha = 1/2\pi + n2\pi$ . Then  $\cos(\omega t + \alpha) = 0$  and thus  $v = 0$ .  
By displacement is 0,  $x = 0$ , and therefore  $\omega t + \alpha = n2\pi$ . Then  $\cos(\omega t + \alpha) = 1$ , and therefore the speed is at its maximum,  $\omega A$ .

The force acting on a particle can be calculated by applying the second law of Newton:

$$F = m * a = m * \frac{dv}{dt} \text{ as } a = \frac{dv}{dt}$$

$$F = -m\omega^2 A * \sin(\omega t + \alpha) = -m\omega^2 x$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad a_{\max} = A\omega^2$$

### Spring

Law of Hooke:  $F = -kx$ . In this formula, k is the elasticity constant.

$$\omega = \sqrt{\frac{k}{m}} ; T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}$$

With a spring constant k two times larger, the oscillation frequency increases by  $\sqrt{2}$ .

$$E_{\text{kin}} = 1/2mv^2 = 1/2m\omega^2 A^2 \cos^2(\omega t + \alpha)$$

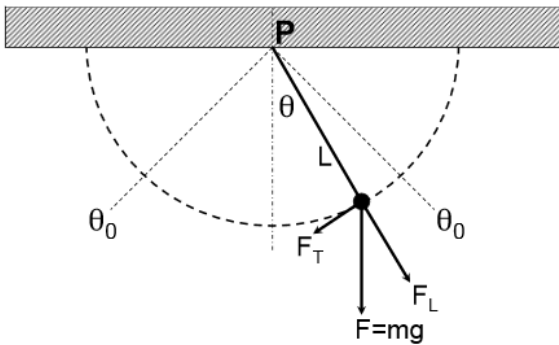
$$E_{\text{pot}} = \int_{x=0}^x -F dx = \int_{x=0}^x m\omega^2 x dx = 1/2m\omega^2 x^2 = 1/2m\omega^2 A^2 \sin^2(\omega t + \alpha)$$

$$E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}} = 1/2m\omega^2 A^2 \quad \text{depends on the elasticity constant and the maximum amplitude}$$

A harmonic oscillator works as an isolated system wherein the total energy remains constant while the distribution of this energy between kinetic and potential energies varies in time.

### Pendulum

A mathematical pendulum is an idealized real pendulum: it is a point mass  $m$  that is suspended from a fixed point  $P$  by means of a mass less rigid wire with length  $L$ .



$$x = L\theta \quad h = L(1 - \cos(\theta))$$

$$F_T = -mg \sin \theta$$

The minus sign in the formula accounts for the fact that the force is opposed to the displacement of the pendulum at that moment.

$$F_T = ma_T = mL \frac{d^2\theta}{dt^2} = -mg \sin \theta \quad (\text{if } \theta \text{ is low, } \sin \theta \approx \theta)$$

$$> \quad \frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

The solution of this differential equation is:  $\theta = \theta_0 \sin(\omega t + \alpha)$  where  $\omega^2 = g/L$ .

The consequence of a bigger  $\theta$  is that increasing amplitudes will cause the period  $T$  of the oscillation to diverge from the above formulas.

The oscillating systems above were not influenced by external forces. In practice there is always friction present. Systems will be driven with a force from outside.

- By giving a push to the object exactly at the maximum displacement, where the speed is 0 such that the object receives a speed  $\Delta v$ ,  $E_{kin} \rightarrow$ . The new kinetic energy is:  $1/2m(v \pm \Delta v)^2$ .
- By giving the object an extra potential energy at the maximum displacement (pull it further out of the equilibrium position),  $E_{pot} \rightarrow$ .

In realistic systems, there is always energy loss. The displacements decrease because of friction. The

friction is proportional but opposite to the speed:  $F_{friction} = -Cv = -C \frac{dx}{dt}$

$$F_h + F_{friction} = -kx - C \frac{dx}{dt} = m \frac{d^2x}{dt^2}$$

$$> \frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = 0 \quad \text{where } \omega_0^2 = \frac{k}{m} \text{ and } 2\gamma = \frac{C}{m}$$

Addition of an external force will add a harmonic force with an angular frequency  $\omega$ :  $F(t) = F_0 \cos(\omega t)$

$$F_h + F_{friction} + F(t) = -kx - C \frac{dx}{dt} + F_0 \cos(\omega t) = m \frac{d^2x}{dt^2}$$

$$> \frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t)$$

$$\text{Solution without friction: } \frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t)$$

$$> x(t) = A \cos(\omega t - \alpha) \quad \text{you can only solve when } \alpha = 0 \text{ or } \alpha = \pi \text{ with } k = m\omega^2$$

$$- \alpha = 0: A = \frac{F_0/m}{\omega_0^2 - \omega^2}$$

$$- \alpha = \pi: A = \frac{F_0/m}{\omega^2 - \omega_0^2}$$

Solution for resonance without friction:  $\omega = \omega_0^2 \rightarrow$  energy is continuously passed from the driving force to the oscillation.

$$x(t) = A(t) \sin(\omega_0 t)$$

We expect a phase difference  $1/2\pi$  between the system and the driving force, this is precisely the difference between a sine and cosine.

$$\frac{F_0}{m} \cos(\omega_0 t) = \frac{d^2 A(t)}{dt^2} \sin(\omega_0 t) + 2 \omega_0 \frac{dA(t)}{dt} \cos(\omega_0 t)$$

This equation can be solved when  $\frac{d^2 A(t)}{dt^2} = 0$  > expression is met later

$$\frac{F_0}{m} \cos(\omega_0 t) = 2 \omega_0 \frac{dA(t)}{dt} \cos(\omega_0 t) \quad \rightarrow \quad \frac{dA(t)}{dt} = \frac{F_0}{2m\omega_0} \quad \rightarrow \quad A(t) = \frac{F_0}{2m\omega_0} t + C$$

$$x = \left( \frac{F_0}{2m\omega_0} t + C \right) \sin(\omega_0 t)$$

Solution with friction:  $\frac{d^2 x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = 0$

$$x = A(t) \sin(\omega t)$$

Results in:

$$A'' \sin(\omega t) + 2\omega A' \cos(\omega t) - A\omega^2 \sin(\omega t) + 2\gamma (A' \sin(\omega t) + \omega A \cos(\omega t)) + A\omega_0^2 \sin(\omega t) = 0$$

$$\sin(\omega t)(A'' - A\omega^2 + 2\gamma A' + A\omega_0^2) = 0 \quad \text{and} \quad \cos(\omega t)(2\omega A' + 2\gamma \omega A) = 0$$

From the cosine function follows:  $A' = -\gamma A \rightarrow A(t) = Be^{-\gamma t}$

From the sine function follows:  $\gamma^2 A - A\omega^2 - 2\gamma^2 A + A\omega_0^2 = 0 \rightarrow \gamma^2 = \omega_0^2 - \omega^2$

$$x(t) = Be^{-\gamma t} \sin(\sqrt{\omega_0^2 - \gamma^2} * t) \quad (\text{weakly damped: } \gamma < \omega_0)$$

$$x(t) = Ae^{-\gamma t} + Bte^{-\gamma t} \quad (\text{critically damped: } \gamma = \omega_0)$$

$$\text{Imaginary numbers} \quad (\text{over-damped: } \gamma > \omega_0)$$

Solution forced, damped oscillation:  $\frac{d^2 x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t)$

$$x = A \sin(\omega t - \alpha)$$

Results in:

$$-A\omega^2 \sin(\omega t - \alpha) + 2\gamma A \omega \cos(\omega t - \alpha) + A \omega_0^2 \sin(\omega t - \alpha) = \frac{F_0}{m} \cos(\omega t)$$

USE:  $\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$  OR:  $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

$$\sin(\omega t) [A \omega_0^2 \cos(\alpha) - A \omega^2 \cos(\alpha) + 2\gamma A \omega \sin(\alpha)] + \cos(\omega t) [A \omega^2 \sin(\alpha) - A \omega_0^2 \sin(\alpha) + 2\gamma A \omega \cos(\alpha) - \frac{F_0}{m}] = 0$$

Both terms in the brackets are constant in time, it is now convenient to divide everything by  $A \cos \alpha$ .

$$\sin(\omega t) [(\omega_0^2 - \omega^2) + 2\gamma \omega \tan(\alpha)] + \cos(\omega t) [(\omega^2 - \omega_0^2) \tan(\alpha) + 2\gamma \omega - \frac{F_0}{A m \cos(\alpha)}] = 0$$

From the sine function follows:  $\tan \alpha = \frac{\omega^2 - \omega_0^2}{2\gamma \omega}$

$$\frac{1}{\cos \alpha} = \sqrt{1 + \tan^2 \alpha} = \sqrt{1 + \frac{(\omega^2 - \omega_0^2)^2}{4\gamma^2 \omega^2}}$$

$$x = \frac{F_0}{m \sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2 \omega^2}} \sin(\omega t - \alpha) \quad \text{with} \quad \tan \alpha = \frac{\omega^2 - \omega_0^2}{2\gamma \omega}$$

Full solution of weakly damped systems:  $x = \frac{F_0}{m \sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2 \omega^2}} \sin(\omega t - \alpha) + Be^{-\gamma t} \sin(\sqrt{\omega_0^2 - \gamma^2} * t)$

The last term will decrease significantly. This means that the amplitude after a long period of time depends on the amplitude of the driving force  $F_0$ .

$$\alpha = a + bi = |\alpha| (\cos \phi + i \sin \phi)$$

$$a = |\alpha| \cos \phi \quad b = |\alpha| \sin \phi \quad |\alpha| = \sqrt{a^2 + b^2}$$

$$e^{ix} = \cos(x) + i \sin(x) \quad \rightarrow f'(x) = i * f(x)$$

Solving the equation for forced, damped oscillations goes in two steps (with imaginary numbers):

1. Find a solution for the complete, inhomogeneous equation, the particular solution.
2. Find a solution to the homogeneous version of the equation, the characteristic solution which provides the start-up phenomena.

$$x > x_0 * e^{i(\omega t - \alpha)}$$

$$\text{Results in: } -\omega^2 x_0 * e^{i(\omega t - \alpha)} + 2\gamma i \omega x_0 * e^{i(\omega t - \alpha)} + \omega_0^2 x_0 * e^{i(\omega t - \alpha)} = \frac{F_0}{m} e^{i(\omega t + \pi/2)}$$

Divide with  $e^{i(\omega t - \alpha)}$ :

$$((\omega_0^2 - \omega^2) + 2i\gamma\omega)x_0 = \frac{F_0}{m} e^{i(\alpha + \pi/2)}$$

Separate in real and imaginary part:

$$(\omega_0^2 - \omega^2)x_0 = \frac{F_0}{m} \cos(\alpha + \pi/2) = -\frac{F_0}{m} \sin(\alpha) \quad \text{AND:} \quad 2\gamma\omega x_0 = \frac{F_0}{m} \cos(\alpha)$$

Divide the left sides with each other:

$$\tan \alpha = \frac{\omega^2 - \omega_0^2}{2\gamma\omega}$$

Returning to  $((\omega_0^2 - \omega^2) + 2i\gamma\omega)x_0 = \frac{F_0}{m} e^{i(\alpha + \pi/2)}$ , multiply with  $e^{-i(\alpha + \pi/2)}$ :

$$((\omega_0^2 - \omega^2) + 2i\gamma\omega)x_0 * (-\sin \alpha - i \cos \alpha) = \frac{F_0}{m}$$

$$\text{Real part: } x_0(-(\omega_0^2 - \omega^2)\sin \alpha + 2\gamma\omega \cos \alpha) = \frac{F_0}{m}$$

$$\cos \alpha = \frac{1}{\sqrt{1 + \tan^2 \alpha}} \quad \sin \alpha = \frac{\tan \alpha}{\sqrt{1 + \tan^2 \alpha}}$$

$$x_0 = \frac{F_0}{m} * \frac{\sqrt{1 + \tan^2 \alpha}}{-(\omega_0^2 - \omega^2)\tan \alpha + 2\gamma\omega} = \frac{F_0}{m} * \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}}$$

$$x > \frac{F_0}{m} * \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}} * e^{i(\omega t - \alpha)}$$

$$x > Ce^{\lambda t}$$

$$\lambda^2 + 2\gamma\lambda + \omega_0^2 = 0$$

$$\lambda = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$x(t) = e^{\lambda_1 t} + e^{\lambda_2 t} \quad \text{OR:} \quad x(t) = e^{-\gamma t} (A_1 e^{\sqrt{\gamma^2 - \omega_0^2} t} + A_2 e^{-\sqrt{\gamma^2 - \omega_0^2} t})$$

- Weakly damped oscillation:  $\gamma < \omega_0$ , the exponents become purely imaginary.
- Over-damped oscillation:  $\gamma > \omega_0$ , the exponents become real.
- Critically damped oscillation:  $\gamma = \omega_0$ ,  $x(t) = Ae^{-\gamma t} + Bte^{-\gamma t}$

The stationary solution is:

1. The motion is harmonic.
2. The angular frequency is  $\omega$  and not  $\omega_0$ .
3. The amplitude of the oscillation is dependent on  $\omega$ .
4. When  $\omega \approx \omega_0$ , then the amplitude is maximal: resonance.
5. The amplitude also depends on the friction constant  $\gamma$ , as  $\gamma$  increases, the maximum amplitude decreases.
6. Small values of  $\gamma \rightarrow$  sharp resonances with large amplitudes.
7. Large values of  $\gamma \rightarrow$  broad resonances with small amplitudes.

## Electricity

Electricity begins with charge. The charge of one electron is  $-1.603 \cdot 10^{-19}$  Coulomb. 1 Coulomb of charge is equivalent to that of  $6,24 \cdot 10^{18}$  electrons.

Matter is in general neutral which means that there must also be positive charge carriers. When looking at conduction, these carriers are unimportant as they are usually immobile. Positive particles are often ions that are fixed in solid material.

In plasmas (gaseous states), both positive and negative charges are able to move. If both types of charge move in the same direction then they have opposite effects on the strength of the current.

Under certain conditions, moving charges can cause a 'current' of charge (electric currents). The unit of current is Coulomb/second (the amount of charge that passes per second) = Ampere. **1A = 1C/s.**

In order for a current to run, there has to be a potential difference: electrons must have less electric energy after they have passed through this potential difference (voltage source). The greater the potential difference, the faster the flow.

$\Delta E_{\text{pot}} = Q \cdot V = \text{amount of charge} \cdot \text{potential difference.}$

A common potential level has to be defined, relative to which other values are measured. This common level is called 'ground': a very good conducting rod which is inserted deep into the ground. This is done because we want to avoid that our electrical circuit or the charge distribution in the object in our setup can change the reference potential. The earth potential will not change.

Ohm's law:  $V = I R.$

The electric current  $I$  is proportional to the potential difference  $V$ . The proportionality constant  $R$  is the resistance that the electrons meet during the route between two points.

$P(\text{ower}) = I V$  (Watts, Joule/second, Volt-Ampere)

The transformed energy per second (dissipated energy) is equal to the reduction of potential energy per second.

$$P = I V = I I R = I^2 R = V^2/R$$

The resistance depends on the type of metal of the wire.

Resistivity or specific resistance ( $\rho$ ): a material constant that describes the resistance of a wire of certain dimensions.

$$R = \rho L / a = \rho L / \pi r^2 (\Omega) \quad \rho = R a / L (\Omega m)$$

Standard resistances exist for electric circuits. These resistances are distinguished from each other according to certain coding.

Voltage sources give a potential difference while resistors decrease the potential difference when they conduct a current.

**Kirchhoff's first law:** the algebraic sum of the amperages of the wires that come together in a hub is 0. The direction of the current in relation to the hub also needs to be taken into account.

**Kirchhoff's second law:** the sum of all potential differences is 0. The sum of all voltage supplies is equal to the sum of the potential drop across each of the resistors in this circuit.

$$\text{In series: } V - IR_1 - IR_2 = 0$$

$$R_s = R_1 + R_2$$

$$V_s = V_1 + V_2$$

$$\text{In parallel: } I_1 R_1 - I_2 R_2 = 0$$

$$1/R_s = 1/R_1 + 1/R_2$$

$$I_s = I_1 + I_2$$

Batteries: the basic principle is that chemical energy in the battery transforms into electric energy. There are always two different electrodes, during the process of current generation, atoms from one electrode are carried over to the other one. The battery consists of the anode, the cathode and the electrolyte.

### Time dependent electricity

Capacitors: electronic components in which a charge can be stored. It basically consists of two electrodes that are separated by an insulator. The charge is stored close to each other, so that the + attractive force compensates the repulsive forces. Special types of capacitors are adjustable in their capacity.

Leakage influences the working of the capacitors. The size and cause of the leakage differs per type of capacitor. The leakage resistance determines how long it will take for the voltage to drop after a device has been charged.

If a charge  $Q$  is stored, a voltage  $V$  is created over the terminals of the capacitor which is proportional to the amount of charge. The capacitor  $C$  is the charge which is needed to obtain a voltage of  $1V$  over this capacitor.

$$Q = C \cdot V \quad (\text{Farad, Coulomb/Volt}) \quad p = 10^{-12} \quad n = 10^{-9} \quad \mu = 10^{-6}$$

In series:  $V_{\text{source}} = V_1 + V_2$        $Q$  is equal over both capacitors       $1/C_v = 1/C_1 + 1/C_2$

In parallel:  $V$  is equal over both capacitors       $Q_{\text{total}} = Q_1 + Q_2$        $C_s = C_1 + C_2$

A circuit with a power supply  $V$ , a resistor  $R$  and a capacitor  $C$ :

$$V = I R + Q / C$$

Differentiate in time:

$$0 = R \frac{dI}{dt} + \frac{1}{C} \frac{dQ}{dt} \quad \frac{dQ}{dt} \text{ is always equal to } I \quad \rightarrow \quad \frac{dI}{dt} = - \frac{1}{RC} I$$

$$I(t) = K * e^{-t/RC}$$

$$I(0) = V/R = K$$

$$I(t) = \frac{V}{R} * e^{-t/RC} \quad \rightarrow \quad V_R = I R = V * e^{-t/RC}$$

The speed of decrease depends on the  $RC$  product: the  $RC$  time in seconds.

$$V_C(t) = \frac{Q(t)}{C} = \frac{1}{C} \int_0^t I(t) dt = -V [e^{-t/RC}]_t/0 = V(1 - e^{-t/RC})$$

$$V = V_C + V_R \quad (\text{this can be verified by the formulas above})$$

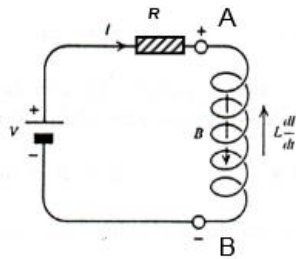
Inductor: a conductive wire that is coiled many times. A coil is a conducting wire which is wound in a circle form, often, around a core of iron or another material. A change of current through such a wound wire has the effect that a force arises which opposes the change (the law of Lenz).

The changing current causes a changing magnetic field. This field creates a voltage in the coil, which is proportional to the speed with which the magnetic field changes and the size of the coil.

$$V_L = -L \frac{dI}{dt}$$

The induction voltage opposes the changes in the current,  $L$  is the self-induction and  $dI/dt$  is the speed with which the current changes.

The unit of self-induction is Henry (H). In a coil of 1 Henry, a voltage of 1 Volt is created for a change of 1 A/s in the current.



Here  $V_L$  is the difference between after ( $V_B$ ) and before ( $V_A$ ).

Suppose that  $I$  increases, then  $-L \frac{dI}{dt}$  is negative  $\rightarrow V_A > V_B$

Suppose that  $I$  decreases, then  $-L \frac{dI}{dt}$  is positive  $\rightarrow V_A < V_B$

After switching on, the current will stay under its final value for a while due to the opposing force in the inductor.

$$V = IR + L \frac{dI}{dt} \quad \rightarrow \quad L \frac{dI}{dt} = V - IR$$

$$Z = V - IR \quad \rightarrow \quad \frac{dZ}{dt} = -R \frac{dI}{dt} \quad \rightarrow \quad \frac{dI}{dt} = -\frac{1}{R} \frac{dZ}{dt}$$

Results in:

$$\frac{dZ}{dt} = \frac{-R}{L} Z$$

$$\rightarrow Z(t) = K e^{-Rt/L}$$

$$I(0) = 0$$

$$\rightarrow I(t) = \frac{Vb}{R} - K e^{-Rt/L}$$

$$\rightarrow I(t) = \frac{Vb}{R} (1 - e^{-Rt/L})$$

$$V_R(t) = R I(t) = V(1 - e^{-Rt/L})$$

$$V_L(t) = V - V_R = V e^{-Rt/L}$$

Turning the voltage on:	C	L
$I(t=0)$	present, maximal	still=0
$V(t=0)$	still=0	present, maximal

The rate of change of the inductor and the resistor is equal to  $L/R$  (s).

The characteristic time for both charging and discharging is the RC-time. The smaller this value, the faster the system reacts.

### Alternating currents

AC circuits: electric circuit with voltage of current supplies which periodically change direction. The voltage of such a source varies harmonically:

$$V(t) = V \sin(\omega t + \alpha) = V \sin(\omega(t+T) + \alpha) \quad \text{with } \omega = 2\pi f \text{ and } T = 1/f = 2\pi/\omega$$

$V$  is the amplitude,  $\omega$  the angular frequency and  $\alpha$  the phase.

Harmonic signals occur very often in nature, because they are projections of a circular movement along one of the axes.

Resistance in an alternating circuit: the current through the resistance has the same phase as the voltage.

$$V(t) = V * e^{i\omega t} (= \cos \omega t + i \sin \omega t)$$

This means that  $V(t) = I(t) R = R I_R e^{i\omega t}$  where  $R = V/I_R$

Capacitor in an alternating circuit:

$$V(t) = V * e^{i\omega t}$$

This means that  $V(t) = V_C = Q/C = \frac{1}{C} \int I(t) dt$

Differentiate with respect to time:

$$i\omega V e^{i\omega t} = \frac{1}{C} I(t) \rightarrow I(t) = i\omega C V(t) \rightarrow I(t) = \omega C V e^{i(\omega t + \pi/2)}$$

Ohm's law in complex notation: impedance, Z. The impedance contains information both about the size as well as the phase of the quantity. The impedance of a capacitor is purely imaginary which indicates that the functions for V and I are 90° shifted with respect to each other.

Inductor in an alternating circuit:

$$V(t) = L \frac{dI}{dt}$$

$$L * I(t) = \int V(t) dt = \int V e^{i\omega t}$$

$$I(t) = \frac{1}{i\omega L} V e^{i\omega t} = \frac{1}{i\omega L} V(t)$$

$$Z = V(t)/I(t) = i\omega L$$

Here also a factor i used, meaning that in the inductor, the current and voltage are 90° shifted.

### Impedance

The impedance Z is always determined by two parameters: a) either by the real and imaginary component of the impedance b) or by the modulus |Z| and the argument  $\phi$ .

Impedance of a resistor:

$$Z = R \quad \rightarrow \quad |Z| = R \quad \phi = 0$$

Impedance of a capacitor:

$$Z = \frac{-i}{\omega C} \quad \rightarrow \quad |Z| = \frac{1}{\omega C} \quad \phi = -1/2\pi$$

Impedance of a coil:

$$Z = i\omega C$$

Impedances in series just add up, and in parallel they follow the law of parallel resistors.

**The RC-filter** in an RC-circuit.

$$V_{in} = V * e^{i\omega t}$$

$$Z_R = R \quad Z_C = -i/\omega C \quad Z = Z_R + Z_C$$

$$|Z| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} \quad \text{and the angle } \phi = \arctan\left(\frac{1}{\omega RC}\right)$$

$$V_{in} = Z * I$$

$$V * e^{i\omega t} = |Z| e^{i\phi} * I$$

$$I = \frac{V}{|Z|} e^{i(\omega t - \phi)}$$

$$V_C = I Z_C$$

$$V_{in} = I (Z_C + Z_R)$$

$$\frac{V_C}{V_{in}} = A_C \text{ (transfer function)} = \frac{Z_C}{Z_C + Z_R} = \frac{\frac{-i}{\omega C}}{\frac{-i}{\omega C} + R}$$

Multiply with complex conjugate:

$$A_C = \frac{1 - iR\omega C}{1 + R^2\omega^2 C^2}$$

$$\text{The amplitude of the ratio: } |A_C| = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

$$\text{The phase difference: } \phi = -\arctan(\omega RC)$$

$$\text{Characteristic point: } \omega RC = 1$$

$$\frac{V_R}{V_{in}} = A_R = \frac{Z_R}{Z_R + Z_C} = \frac{(\omega RC)^2 + i\omega RC}{1 + (\omega RC)^2} \quad |A_R| = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} \quad \phi = \arctan\left(\frac{1}{\omega RC}\right)$$



### The RL-filters in an RL circuit

$$Z = Z_R + Z_L = R + i\omega L = |Z| e^{i\phi}$$

$$|Z| = \sqrt{R^2 + \omega^2 L^2} \quad \phi = \arctan\left(\frac{\omega L}{R}\right)$$

$$I = \frac{V}{|Z|} e^{i(\omega t - \phi)}$$

$$A_L = \frac{V_L}{V_{in}} = \frac{Z_L}{Z_L + Z_R} = \frac{i\omega L R + \omega^2 L^2}{R^2 + \omega^2 L^2} \quad |A_L| = \frac{\frac{\omega L}{R}}{\sqrt{1 + \left(\frac{\omega L}{R}\right)^2}} \quad \phi = \arctan\left(\frac{R}{\omega L}\right)$$

If  $R/L = RC$  than both filters are identical.

$$A_R = \frac{R^2 - i\omega L R}{R^2 + \omega^2 L^2} \quad |A_R| = \frac{1}{\sqrt{1 + \left(\frac{\omega L}{R}\right)^2}} \quad \phi = \arctan\left(\frac{\omega L}{R}\right)$$

### Decibels

The decibel is a relative scale for comparing aspects in a wide range of orders of magnitude, such as sound level, but also the strength of electric signals.

$$10 \cdot 10 \log(\text{ratio}) = \text{difference in dB}$$

### LCR resonance circuit

$$Z = R + i\omega L - \frac{i}{\omega C}$$

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

This has a minimum (maximal amplitude) when  $\omega = \sqrt{\frac{1}{LC}}$

$$\phi = \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) \text{ for } \omega = \sqrt{\frac{1}{LC}}$$

$$A_C = \frac{Z_C}{Z_C + Z_L + Z_R} = \frac{\frac{1}{\omega C} \left( \frac{1}{\omega C} - \omega L \right) - iR}{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}$$

$$|A_C| = \frac{\frac{1}{\omega C}}{\sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}}$$

$$|A_L| = \frac{\omega L}{\sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}}$$

$$|A_R| = \frac{R}{\sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}}$$

With the voltage supply turned off:

$$V_R + V_C + V_L = 0$$

$$I R + Q/C + L \, dI/dt = 0$$

$$\rightarrow \frac{d^2 I}{dt^2} + \frac{R}{L} \frac{dI}{dt} + \frac{1}{LC} I = 0$$

Similarities with mechanical equations!

$$\omega_0 = 1/\sqrt{LC}$$

$$\gamma = R/2L$$

$$\text{Weak damping: } \omega_0 > \gamma = R/2L < 1/\sqrt{LC} = R/2 \cdot \sqrt{\frac{L}{C}}$$

With voltage supply:

$$V_R + V_C + V_L = V_{in}$$

$$V_{in} = V_0 \sin(\omega t)$$

$$\rightarrow \frac{d^2 I}{dt^2} + \frac{R}{L} \frac{dI}{dt} + \frac{1}{LC} I = \frac{V_0 \omega}{L} \cos(\omega t)$$

Use the formula provided in the mechanical equations and change the coefficients:

$$I(t) = \frac{V_0 \omega}{L \sqrt{(\omega^2 - \omega_0^2)^2 + \left(\frac{R}{L}\right)^2 \omega^2}} \sin(\omega t - \alpha)$$

$$\text{Phase between the current and the voltage: } \tan \alpha = \frac{(\omega^2 - \omega_0^2)L}{R\omega}$$

$$\omega_{RES} = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2} = \frac{\sqrt{\frac{L}{C} - \frac{R^2}{4}}}{L}$$

When  $\omega_0 = \frac{1}{\sqrt{LC}}$ , the voltage and the current have the same phase which means that the system is quasi-real (phase-resonance).

Energy in mechanical systems:

$$E_{kin} = \frac{1}{2} m v^2 = \frac{1}{2} \left(\frac{dx}{dt}\right)^2$$

$$E_{pot} = \int -F dx = \int_0^x kx dx = \frac{1}{2} m \omega^2 x^2$$

Energy in electrical systems:

- Energy in capacitor

$$V = Q/C$$

$$dW = dQ V$$

$$\rightarrow W = \int dW = \int V dQ = 1/C \int Q dQ = \frac{1}{2} Q^2/C$$

$$> W = \frac{1}{2} C V_c^2$$

- Energy in inductor

$$V = -L (dI/dt)$$

$$dW = dQ V = dQ/dt * V * dt = I * V * dt = I * L * (dI/dt) * dt = L * I * dI$$

$$\rightarrow W = \int dW = \int L I dI = \frac{1}{2} L I^2$$

The total energy in the circuit moves from the inductor to the capacitor and vice versa every  $\frac{1}{4}$  period. The resistor in the circuit converts electrical energy into heat, so this is where energy is lost from the circuit. If the lost energy is not replenished, the amplitude of the current in the circuit will decrease with every oscillation.

Energy transformed in heat per oscillation:

$$P = I^2 R = I_{max}^2 \cos^2(\omega t) R \quad \text{with } I_{max} \text{ as the current amplitude}$$

Integrating over one cycle of oscillation:

$$W = \int_0^{2\pi/\omega} I_{max}^2 \cos^2(\omega t) * R dt = \int I_{max}^2 \cos^2(\omega t) * R dt = \frac{1}{2} I_{max}^2 R * \frac{2\pi}{\omega}$$

The relative importance of the energy loss. Take the moment that all the energy is present in the inductor:  $W_{tot} = \frac{1}{2} L I_{max}^2$

$$Q = 2\pi \frac{\text{stored energy}}{\text{dissipated energy per cycle}} = 2\pi \frac{\frac{1}{2} L I_{max}^2}{\frac{1}{2} R I_{max}^2} = \frac{\omega L}{R}$$

Q is the quality factor of the oscillatory circuit. R/L is damping = 2γ in mechanical circuits.

Expression for the current (more important than transfer across the components):

$$I(t) = \frac{V_0 \omega}{L \sqrt{(\omega^2 - \omega_0^2)^2 + \left(\frac{R}{L}\right)^2 \omega^2}} \sin(\omega t - \alpha)$$

$$\text{Amplitude } I_0 = \frac{V_0}{\sqrt{\frac{L^2}{\omega^2} (\omega^2 - \omega_0^2)^2 + R^2}}$$

The current amplitude is maximum if  $\omega = \omega_0$

$$I_{\max} = I_0(\omega_0) = V_0/R$$

Critical frequency: the value  $\omega$  for which  $I_0$  has decreased to  $1/\sqrt{2} * I_{\max}$

$$\text{Then the term under the square root must be doubled: } \frac{L}{\omega} (\omega^2 - \omega_0^2) = \frac{L(\omega - \omega_0)(\omega + \omega_0)}{\omega} = R$$

You can solve this but an approximate approach gives more insight. This approximation is based on the fact that the bandwidth of the resonance peak is small relative to the value of  $\omega_0$ .

$$\rightarrow \frac{\omega + \omega_0}{\omega} \approx 2; \quad \omega - \omega_0 = \Delta \omega$$

$$> R = 2L\Delta\omega \quad \rightarrow 2\Delta\omega = R/L$$

$$\frac{2\Delta\omega}{\omega_0} = \frac{R}{L \omega_0} = \frac{1}{Q}$$

So the quality factor is directly related to the relative width of the resonance.

## Systems dynamics

Systems engineering: an interdisciplinary field of engineering that focuses on how to design and manage complex engineering systems over their life cycles.

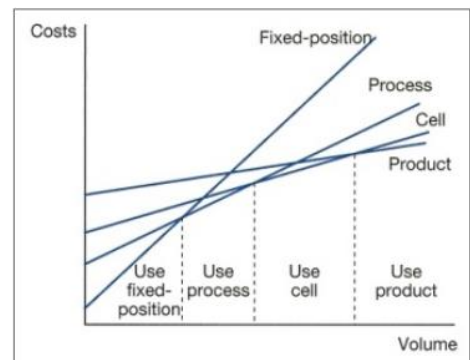
### Important aspects:

1. Context & relations
2. Zoom out
3. Loop effects
4. Time delays
5. Dynamics
6. Patterns

Goal: gaining influence on the system

A system is a dynamic and complex whole, interacting as a structured functional unit. In general one studies material, information and energy flow.

Production organizations can be classified by for example: execution (manual or by machine), process (single or combined), logistics, process type and layout.



Work In Progress (WIP): number of products in the system.  
Batch size: multiple products in a batch that are processed at the same time.

Cycle time / throughput time (CT): the time between the product entering and exiting the system.

Throughput rate (TH): the number of products that leave the system per time unit (heartbeat).

Bottleneck: process that takes the longest time.

Idle time: time a machine is not used (a station other than the bottleneck).

Utilization of a workstation: fraction of time the workstation is not idle (includes machine failure or set-up time).

Balancing: changing the workload of the stations in order to evenly distribute the work.

Lead time: the time allotted for production.

Bottleneck rate,  $r_b$ : the rate of the workstation having the highest utilization.

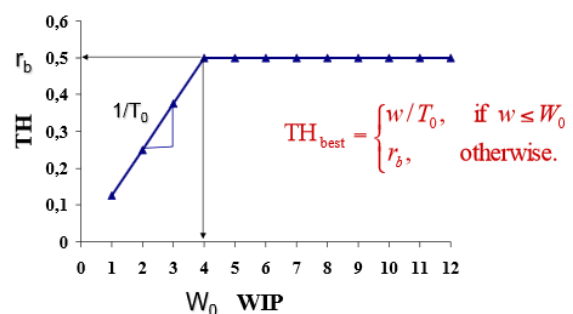
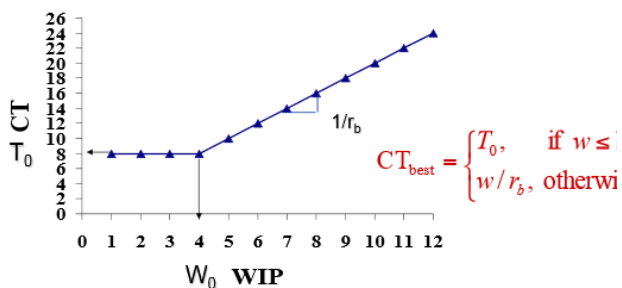
CONWIP: the WIP on a production line is kept constant.

A production line contains similarities with a DC circuit.

Kirchoff's first law:  $\sum I = 0 \rightarrow$  the sum of jobs at a branching point should be 0.

Ohm's law:  $V = I R \rightarrow$  Little's law:  $WIP = \text{throughput rate} * \text{cycle time}$

When there is more WIP than the number of machines on the production line, the cycle time will increase due to waiting times while the throughput will remain the same.



$CT_{best}$  is the Best Case Performance. The turning point in the graph is called the critical WIP ( $W_0$ ).  
 $TH_{best}$  is the Best Case Performance. The turning point is again the critical WIP.  
 $w$  : the WIP applied                       $T_0$ : raw processing time.

Worst case performance: the products are treated as a batch.

$$CT_{worst} = WIP * T_0$$

$$TH_{worst} = 1/T_0$$

Factories use batching in order to avoid changeovers and long set-up times on machines.

In practice the performance of a factory will lie between the Best and Worst case performance: the Practical Worst Case.

Maximum randomness: every possible state occurs with equal frequency.

1. All stations have the same average processing times (well balanced line).
2. All stations consist of single machines.
3. The processing time follows exponential distribution.

We can write down states in terms of vectors, every state is equally likely to occur:

$$(3,0,0,0) \quad (0,3,0,0) \quad \rightarrow \quad (1,0,1,1) \quad (0,1,1,1) \quad \text{spread out}$$

$T_0$  = number of stations \* average production time =  $N t$

$r_b = 1/\text{average production time} = 1/t$

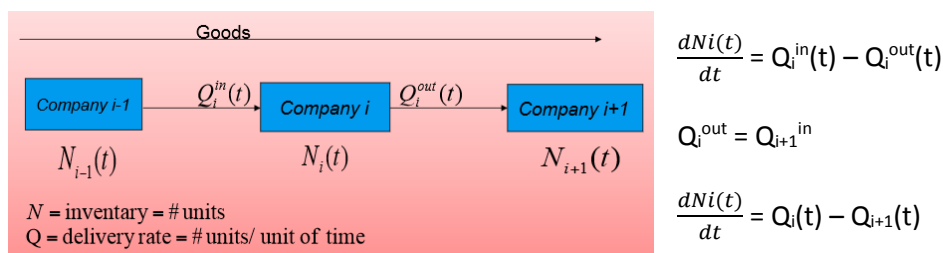
$w$ : number of jobs

$$CT_{PWC}(\text{machine}) = t + \frac{w-1}{N} t$$

$$CT_{PWC}(\text{line}) = Nt + (w-1)t = T_0 + \frac{w-1}{rb}$$

$$TH_{PWC} = \frac{WIP}{CT_{PWC}} = \frac{w}{T_0 + (w-1)/rb} = \frac{w * rb}{W_0 + w - 1} \quad (T_0 = W_0/r_b)$$

A supply chain is the entire network entities serving the same consumer of customer. Producers compete with each other only through their supply chains.



Instabilities in a supply chain often take the shape of oscillations in demand. The bullwhip effect refers to a trend of larger and larger swings in inventory in response to changes in demand, as one looks at companies further back in the supply chain of a product.

Sources of bullwhip effect:

- Demand forecasting
- Order batching
- Long delivery times
- Price fluctuations
- Promotions
- Demand interpretation

Trial function from management:

$$\frac{dQi(t)}{dt} = \frac{1}{T} \left( \frac{Ni^0 - Ni(t)}{\tau} - \beta \frac{dNi}{dt} + \epsilon(Q_i^0 - Q_i(t)) \right)$$

Where:

$N_i^0$  = steady-state inventory

$Q_i^0$  = steady-state delivery rate

T = interval time over which a full oscillation takes place

$\tau, \beta, \epsilon$  = constants

$$n_i(t) = N_i(t) - N_i^0$$

$$q_i(t) = Q_i(t) - Q_i^0$$

$$\frac{dNi(t)}{dt} = Q_i(t) - Q_{i+1}(t)$$

Substitute:

$$\frac{dq_i(t)}{dt} = \frac{1}{T} \left( \frac{-ni(t)}{\tau} - \beta \frac{dni}{dt} + \epsilon \cdot q_i(t) \right)$$

$$\frac{dni(t)}{dt} = q_i(t) - q_{i+1}(t)$$

Substitute:

$$\frac{dq_i(t)}{dt} = -\frac{1}{T} \left( \frac{ni(t)}{\tau} + \beta (q_i(t) - q_{i+1}(t)) + \epsilon \cdot q_i(t) \right)$$

Differentiate:

$$\frac{d^2 qi(t)}{dt^2} = -\frac{1}{T} \left( \frac{1}{\tau} \frac{dni(t)}{dt} + \beta \left( \frac{dq_i(t)}{dt} - \frac{dq_{i+1}(t)}{dt} \right) + \epsilon \frac{dq_i(t)}{dt} \right)$$

Again substitute and simplify:

$$\frac{d^2 qi(t)}{dt^2} + \frac{\beta + \epsilon}{T} \frac{dq_i(t)}{dt} + \frac{1}{T\tau} q_i(t) = \frac{1}{T} \left( \frac{1}{\tau} q_{i+1}(t) + \beta \frac{dq_{i+1}(t)}{dt} \right)$$

We see that the system is driven by customers.

Implies that:

$$\gamma = \frac{\beta + \epsilon}{2T}$$

$$\omega_0 = 1/\sqrt{T\tau}$$

$$\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} = -\frac{1}{2T} ((\beta + \epsilon) \pm \sqrt{(\beta + \epsilon)^2 - \frac{4T}{\tau}})$$

In order to have damping, the first term should be negative:  $(\beta + \epsilon) > 0$ .

In business it is critical that damping takes place. Instabilities should quickly be suppressed.

Weakly damped: bullwhip effect.

## Lee

Companies need to understand what creates the bullwhip effect in order to counteract it.

1. Demand forecast updating: long lead times > great bullwhip effect due to safety stocks.
2. Order batching: periodic ordering amplifies variability.
3. Price fluctuation: the buying pattern does not reflect the consumption pattern.
4. Rationing and shortage gaming: orders give little information about the real demand.

Solutions:

1. Information sharing: demand information at a downstream site is transmitted upstream in a timely fashion.
2. Channel alignment: the coordination of pricing, transportation, inventory planning, and ownership between the upstream and downstream site in a supply chain. Provides incentives for customers to adjust order size and period according to suppliers preferences.
3. Operational efficiency: activities that improve performance such as lead-time reduction.

Distorted information: the lack of knowledge on orders placed in the supply chain. This is due to bad forecasting and unknown backlogs. Not communicating with parties further in the supply chain causes uncertainty in ordering. When a customer in the supply chain runs out of stock, a backlog can form. This backlog adds to the ordering in the next order, resulting unstable ordering behavior.