

Final Examination of Physical Systems

April 4, 2014, 14:00-17:00

INSTRUCTIONS (READ THIS CAREFULLY)

1. **Make sure you solve each problem on a separate sheet.**
2. Write your name and student number clearly on top of every sheet.
3. Solutions which are not on the right sheet will not be graded. If you need extra paper, use the sheets provided and indicate for which problem the extra sheet is used for.
4. Solve the problems in a systematic way and check your answers. If you think you have made a mistake in a calculation, indicate this. Argue then how you intended to get the right answer and write this down.
5. This is a closed-book examination. No books, notes or graphical calculators may be used during the examination.
6. **Write in a legible manner. Unreadable text will not be handled during grading.**

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Problem #1. Complex Numbers and Differential Equations (10 Points)

Consider the differential equation:

$$\frac{d^2x(t)}{dt^2} + a\frac{dx}{dt} + bx = 0$$

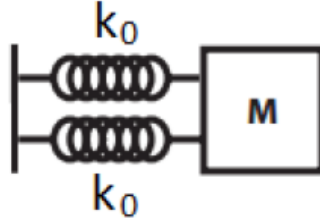
- a) The solution to this differential equation can be written as $x(t) = Be^{\lambda t}$. Use this expression to find λ in terms of a and b . Give the most general solution for $x(t)$. **(3 Points)**
- b) For which values of a and b is λ a complex number? **(2 Points)**

Now take $a = 2$ and $b = 2$ and we specify the initial conditions as $x(0) = 2$ and $x'(0) = 2$.

- c) Determine the two integration constants in the general solution for $x(t)$. **(2 Points)**
- d) For these values of a and b , λ is a complex number and $x(t)$ is a complex function, however, if we write the function $x(t)$ as $C(t) + iD(t)$, the imaginary part D vanishes. Show this explicitly and determine the real part $C(t)$. **(3 Points)**

Problem #2. Mechanical Oscillations without Damping (17 Points)

A body with mass m is connected to two horizontal springs as shown in the figure. The two springs have the same spring constant k_0 . The body moves harmonically with an amplitude A . We consider the friction to be negligible (no damping).



- a) We want to replace the two springs with a single spring. We can choose among three different springs:
- 1) One spring with spring constant $k = k_0$
 - 2) One spring with spring constant $k = \frac{1}{2}k_0$
 - 3) One spring with spring constant $k = 2k_0$

Which spring do we have to choose if we don't want to change the frequency of the harmonic motion? Motivate the answer. Hint: Think about the equivalent spring which could replace these two. **(3 Points)**

- b) What is the differential equation that is appropriate for the motion? Hint: consider the system after replacing the two springs and use k in your answer. **(4 Points)**
- c) Show that $x(t) = A \sin(\omega t)$ is a solution of the equation of motion. What is the expression of the angular frequency ω ? **(4 Points)**

Consider $m = 3 \text{ kg}$, $k = 50 \text{ N/m}$, $A = 30 \text{ cm}$.

- d) What are the maximum values of the speed and of the acceleration of the mass? In which positions does the mass have its maximum speed and acceleration? **(2 Points)**
- e) Give the expression of the total energy of the system and show that it is constant in time. **(4 Points)**

Problem #3. Mechanical Oscillations with Damping (17 Points)

The full differential equation for an externally driven, damped oscillation is given by:

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2x = \left(\frac{F_0}{m}\right)\cos(\omega t)$$

The stationary solution is given by:

$$x_{stat}(t) = \frac{\left(\frac{F_0}{m}\right)}{\sqrt{(\omega^2 - \omega_0^2)^2 + (2\omega\gamma)^2}}\cos(\omega t)$$

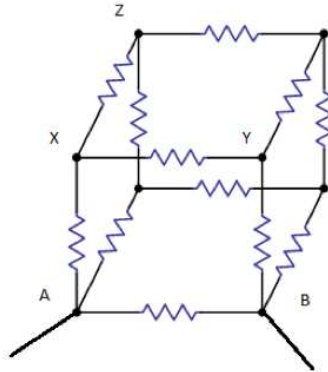
and the transient solution:

$$x_{trans}(t) = Ae^{-\gamma t}\cos\left(\sqrt{\omega_0^2 - \gamma^2}t\right)$$

- Give the homogeneous and inhomogeneous differential equations for this system and indicate to which solution given above (transient or stationary) they correspond. **(4 Points)**
- What is the full solution for an externally driven, damped oscillation? **(3 Points)**
- Derive at which driving frequency (ω) the stationary amplitude is the largest. **(5 Points)**
- Give the expression for the kinetic energy at $t = 0$ for the transient solution. **(5 Points)**

Problem #4. Electric Circuit – DC (18 Points)

Every resistance in the figure below has a resistance of R ohms.

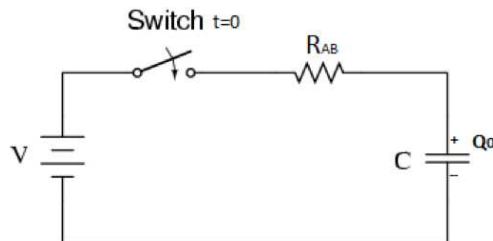


- a) Determine the equivalent resistance R_{AB} of the cube between nodes A & B . **(4 Points)**

A DC power supply ($= V$) is now connected between nodes A & B .

- b) Calculate the current in branch XY . **(2 Points)**
 c) Calculate the current in branch XZ . **(2 Points)**

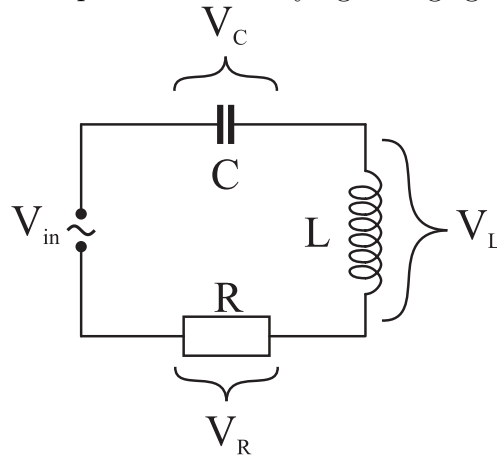
Now, a capacitor ($= C$) initially (time $t = 0$) charged to Q_0 is connected in series with the DC power supply ($= V$) and the resistor cube (R_{AB}); see the figure below.



- d) Determine the expression for charge stored in the capacitor as a function of time. (Hint: Apply Kirchhoffs 2nd law or voltage law.) **(6 Points)**
 e) What is the time constant or characteristic time, τ_C ? Find the current in the circuit when $t = \tau_C$. **(4 Points)**

Problem #5. Electric Circuit – AC (18 Points)

The circuit below is a LCR -circuit which is driven by an alternating voltage V_{in} . This is a driven, dampened system. Alternating voltages with magnitudes V_C , V_L and V_R appear on the capacitor, coil and resistor, respectively. These magnitudes are frequency dependent. The input is a time varying voltage given by: $V_{in} = V_0 \cos(\omega t)$.



- Write down explicitly what the complex impedances are for the capacitor (C) and the coil (L). **(2 Points)**
- Give the equation for the total complex impedance Z_{tot} . **(2 Points)**
- Give the equation for the magnitude of the voltage transfer to the resistor, $|A_R| = |V_R/V_{in}|$ and also the capacitor, $|A_C| = |V_C/V_{in}|$. Make sure that your answers are frequency dependent. **(5 Points)**

The circuit is driven at the eigen-frequency of $\omega = \omega_0 = \sqrt{\frac{1}{LC}}$.

- Under these conditions, explain: What is the phase of the current relative to the driving voltage? Use the magnitudes of the complex impedances of the circuit components in your answer. **(4 Points)**
- The inhomogeneous differential equation for this system is:

$$\underbrace{\frac{d^2 I}{dt^2}}_{Term1} + \underbrace{\frac{R}{L} \frac{dI}{dt}}_{Term2} + \underbrace{\frac{1}{LC} I}_{Term3} = \underbrace{\frac{V_0 \omega}{L} \cos(\omega t)}_{Term4}$$

For the following, choose a term in the equation (Term 1, Term 2, Term 3 or Term 4):

- Which term is related to damping? **(1 Point)**
- Which term is related to the driving force? **(1 Point)**
- Which term describes the restoring force (springiness)? **(1 Point)**

f) The Q -Factor is defined as

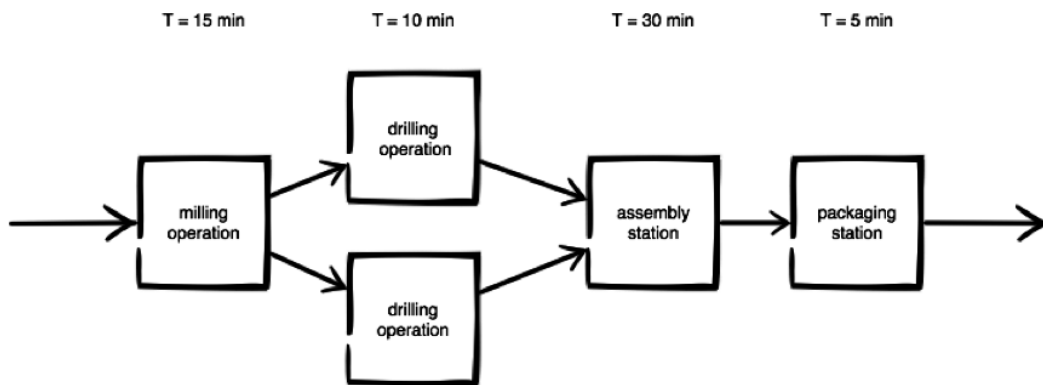
$$Q = 2\pi \frac{\text{Stored Energy}}{\text{Amount of energy dissipated per cycle}}$$

The component values are: $C = 100$ nF, $L = 100$ mH and $R = 10 \Omega$. The driving frequency is set to $\omega = \omega_0$ which results in maximum power dissipation in the resistor. Given that $Q = 100$ and that the driving voltage is $V_0 = 1.0$ V, calculate the amount of Stored Energy? **(2 Points)**

Problem #6. Business Dynamics (20 Points)

You are mostly asked to give mathematical or quantitative answers. Restrict your answer to that. If a definition or description is required, be exact and to the point. Note: incorrect statements in a description will lead to a reduction of the number of points granted.

Consider a production line that looks as follows:



The processing times (T) per station are stated above the stations. Assume one-piece-flow production.

- Give the definition of bottleneck rate. Calculate the bottleneck rate, r_b of the system. Which station is the bottleneck? **(1.5 Points)**
- An important performance measure of a production system is the throughput. Give the definition of throughput. There is a mathematical relation between the throughput of the system and the work-in-progress. Give the formula. Calculate what level of WIP this system needs to perform as a best-case scenario (assume that the cycle time is equal to the raw processing time). **(1.5 Points)**
- In practice, the best-case scenario is often optimistic. Therefore, the practical worst case (PWC) is often used to determine the most likely performance of the system. Give the formula for the PWC throughput as a function of the actual work-in-progress, w on the production floor. Calculate the throughput of the above system for the practical worst case when WIP is set at 3. Why isn't WIP always maximized? Now also draw the curve of the throughput as a function of the actual work-in-process on the production floor. **(3 Points)**
- The production line in the example stated above does not run for 24 hours a day. So at the end of the day, all products still in process are finished and the line is shut down. The next day the line needs to be started up again.

When looking at the performance of a line, we do not want to start measuring its performance right after start up. What is the phase of the production line called after its initial phase? Why do we measure performance after this initial state? Draw the production line at $t = 25$ min, $t = 35$ min, $t = 60$ min and indicate on which times the line is still in its initial state. Assume a WIP of 3. **(5 Points)**

- e) Considering the performance of a production line, there is such a thing as worst-case performance. This forms a baseline to indicate the performance a production line should at least meet. Calculate the throughput worst performance of this production line to show its minimum performance. What causes can occur when a production line performs below this baseline? **(2 Points)**
- f) As stated in the reader, human behavior is aimed at improving the performance of a production line. However, human behavior can also be counterproductive when studying a supply chain instead of a production line. Indicate why this is and what the role of control is. **(2 Points)**
- g) Oscillation effects in a supply chain can be damped. Which of the three types of damping is desirable in a supply chain? Why? Is it possible to achieve this in practice? **(2 Points)**
- h) The differential equation for this supply chain is given by:

$$\frac{d^2 q_i}{dt^2} + \frac{(\zeta + \delta)}{T} \frac{dq_i}{dt} + \frac{1}{T \cdot \tau} q_i(t) = \frac{1}{T} \left(\frac{q_{i+1}(t)}{\tau} + \zeta \frac{dq_{i+1}}{dt} \right)$$

Determine the dampening factor, eigen frequency and driving force of the system. **(3 Points)**

**ANSWERS TO QUESTIONS OF THE EXAMINATION OF
PHYSICAL SYSTEMS, APRIL 4, 2014**

PROBLEM 1

- (a) Plugging in the trial function $x(t) = Be^{\lambda t}$ gives the characteristic equation:

$$\lambda^2 + a\lambda + b = 0, \quad (2)$$

The solution is

$$\lambda_{\pm} = \frac{1}{2} \left(-a \pm \sqrt{a^2 - 4b} \right). \quad (3)$$

This makes the most general solution to the differential equation

$$x(t) = B_+ e^{\lambda_+ t} + B_- e^{\lambda_- t}. \quad (4)$$

- (b) Both λ 's are complex when $a^2 - 4b < 0$.

- (c) When $a = 2$ and $b = 2$, we have $\lambda_{\pm} = -1 \pm i$. So

$$\begin{aligned} x(0) &= B_+ + B_- = 2 \\ x'(0) &= \lambda_+ B_+ + \lambda_- B_- = (-1 + i)B_+ + (-1 - i)B_- = 2 \\ &\text{combine to give } B_- = 1 + 2i \text{ and } B_+ = 1 - 2i. \end{aligned} \quad (5)$$

- (d) First write:

$$\begin{aligned} x(t) &= B_+ e^{(-1+i)t} + B_- e^{(-1-i)t} \\ &= B_+ e^{-t} (\cos(t) + i \sin(t)) + B_- e^{-t} (\cos(-t) + i \sin(-t)) \\ &= (B_+ + B_-) e^{-t} \cos(t) + i(B_+ - B_-) e^{-t} \sin(t). \end{aligned}$$

then fill in B_{\pm} to find that

$$C = 2e^{-t} \cos(t) + 4e^{-t} \sin(t), \quad D = 0. \quad (6)$$

Problem 2

- a) The springs are connected in parallel, so the equivalent constant is

$$k_{\text{eq}} = k_0 + k_0 = 2k_0$$

So we have to choose the spring with constant $k = 2k_0$.

b) $\frac{d^2 x}{dt^2} + \frac{k}{m} x = 0$

c) $\omega = \sqrt{\frac{k}{m}}$

To show that $x(t) = A\sin(\omega t)$ is a solution they have to take it, and its second derivative, and put them into the equation to obtain $0 = 0$

- d) Speed: it is the first derivative of $x(t)$, therefore $v(t) = A\omega\cos(\omega t)$. The maximum speed is obtained every time that $\cos(\omega t) = \pm 1$, that is, at the equilibrium point of the motion. Its absolute value will be

$$V_{\max} = \omega A = 1.2 \text{ m/s}$$

The acceleration is the second derivative of $x(t)$, therefore $a(t) = A\omega^2\sin(\omega t)$. Its maximum value, for $\sin(\omega t) = \pm 1$, is obtained at the extremes of the motion, and is (in modulus)

$$a_{\max} = A\omega^2 = 5 \text{ m/s}^2$$

- e) $E_{\text{kin}} = \frac{1}{2} mv^2 = \frac{1}{2} mA^2\omega^2\cos^2(\omega t)$

$$E_{\text{pot}} = \frac{1}{2} kx^2 = \frac{1}{2} kA^2\sin^2(\omega t)$$

$$E_{\text{tot}} = \dots = \frac{1}{2} kA^2$$

$$\frac{dE_{\text{tot}}}{dt} = 0. \text{ So, the energy is constant.}$$

Problem 3

- a) The homogeneous equation

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2x = 0$$

gives the transient solution. The inhomogeneous equation:

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2x = \frac{F_0}{m}\cos\omega t$$

yields the stationary solution.

- b) $x_{\text{stat}}(t) + x_{\text{trans}}(t)$

- c) Find the maximum of

$$A = \frac{F_0}{m\sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2}}$$

or minimum of $(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2$. Differentiate with respect to ω , set it equal to 0 and obtain the final answer: $\omega = \sqrt{\omega_0^2 - 2\gamma^2}$.

d) $E_{kin} = 1/2mv^2$

$$v = \frac{dx}{dt} = Ae^{-\gamma t} \cdot (-\gamma) \cdot \cos(\sqrt{\omega_0^2 - \gamma^2}t) + Ae^{-\gamma t} \cdot (-\sqrt{\omega_0^2 - \gamma^2}) \cdot \sin(\sqrt{\omega_0^2 - \gamma^2}t)$$

So, at $t=0$, $v=-A\gamma$ and $E_{kin}=1/2m(A\gamma)^2$.

Problem 4

- a) First try to see where (at which cube points), the voltages are the same. When you find these points, you can then make 12 resistors in the plane. For instance, the resistor directly between A and B form one resistor in parallel with the rest. Amongst the rest (except one which is diagonally across AB), you should see how they couple two resistors at the same time (due to the fact that they are at the same potential) and you should put them properly in series or parallel to obtain the equivalent resistor: $R_{AB} = \frac{7}{12}R$.

b) $I_{XY} = \left(\frac{2}{7}\right)\left(\frac{V}{R}\right)$

c) $I_{XZ} = \left(\frac{1}{14}\right)\left(\frac{V}{R}\right)$

d) $Q = CV - (CV - Q_0)\exp\left(-\frac{t}{R_{AB}C}\right)$

e) $\tau_C = R_{AB}C, I(t = \tau_C) = \frac{CV - Q_0}{RC} \left(\frac{1}{e}\right)$

Problem 5

a) $Z_C = \frac{-i}{\omega C} \quad Z_L = i\omega L \quad Z_R = R$

b) $Z_{tot} = Z_C + Z_L + Z_R = \frac{-i}{\omega C} + i\omega L + R$

c) $|A_R| = \left|\frac{V_R}{V_{in}}\right| = \left|\frac{Z_R}{Z}\right| = \left|\frac{R}{\frac{-i}{\omega C} + i\omega L + R}\right| = \frac{1}{\sqrt{\left(\omega \frac{L}{R} - \frac{1}{\omega RC}\right)^2 + 1}}$

$$|A_C| = \left|\frac{V_C}{V_{in}}\right| = \left|\frac{Z_C}{Z}\right| = \left|\frac{-i/\omega C}{\frac{-i}{\omega C} + i\omega L + R}\right| = \frac{1}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}}$$

- d) At a frequency of $\omega = \omega_0 = \sqrt{\frac{1}{LC}}$, we have current resonance and phase resonance, meaning that the current is exactly in sync with the driving voltage.

Also: putting the condition for $\omega = \omega_0$ into $\frac{Z_R}{Z} = \frac{R}{\frac{-i}{\omega C} + i\omega L + R}$ gives :

$$a. \frac{Z_R}{Z} = \frac{R}{\frac{-i}{\sqrt{\frac{1}{LC}}C} + i\sqrt{\frac{1}{LC}}L + R} = \frac{R}{-i\sqrt{\frac{L}{C}} + i\sqrt{\frac{L}{C}} + R} = 1$$

- b. Meaning that the voltage across the resistor is exactly the same in both amplitude and time as the driving voltage, so the phase difference is zero.

- c. Basically, the sum of the impedances of the capacitor and the inductor is

$$\frac{-i}{\omega C} + i\omega L. \text{ If you put the resonance frequency in there you get } -i\sqrt{\frac{L}{C}} +$$

$$i\sqrt{\frac{L}{C}}, \text{ which is zero. Since the current is determined ONLY by the total}$$

impedance the current behaves exactly the same as in a system with only a resistor: eg, you get the same current if you remove both the capacitor and the coil.

- e) Short description of answers:

- Damping \rightarrow Term2, the only term that contains a resistance, also it contains a first derivative, which is analogous to 'speed'
- Driving force \rightarrow Term4, the driving force is the part that makes the differential equation inhomogeneous. Also, V_0 is clearly listed.
- Restoring force \rightarrow Term3, restoring force is proportional to deviation, eg, Hooks law.

- f) The time period of one cycle is $T = \frac{2\pi}{\omega}$

If we drive at the resonance frequency the power dissipated in the resistor is:

$$P = \frac{U^2}{R} = \frac{V_{0RMS}^2}{R}$$

The total energy dissipated in one cycle is:

$$E_{cycle} = P * T = \frac{V_{0RMS}^2}{R} * \frac{2\pi}{\omega}$$

$$\text{So, the total energy stored is: } E_{stored} = \frac{E_{cycle} * Q}{2\pi} = \frac{\frac{V_{0RMS}^2}{R} * \frac{2\pi}{\omega} * Q}{2\pi} = \frac{V_{0RMS}^2}{R} * \frac{1}{\omega} * Q = \frac{Q\sqrt{LC}V_{0RMS}^2}{R}$$

$$E_{stored} = \frac{100 * \sqrt{100 * 10^{-9} * 100 * 10^{-3}} * 1}{10} = \frac{100 * \sqrt{10^{-8}}}{10} = \frac{100 * 10^{-4}}{10} = 1 * 10^{-3} \text{ J}$$

Problem 6

- a) Bottleneck rate, r_b , is the rate (jobs per unit of time) of the workstation having the highest utilization
2 products/ hour

Station 3, assembly

- b) Throughput TH – is the average output of the production line per unit of time.
 $WIP = TH \times CT$

$$CT = T_0 = 1h$$

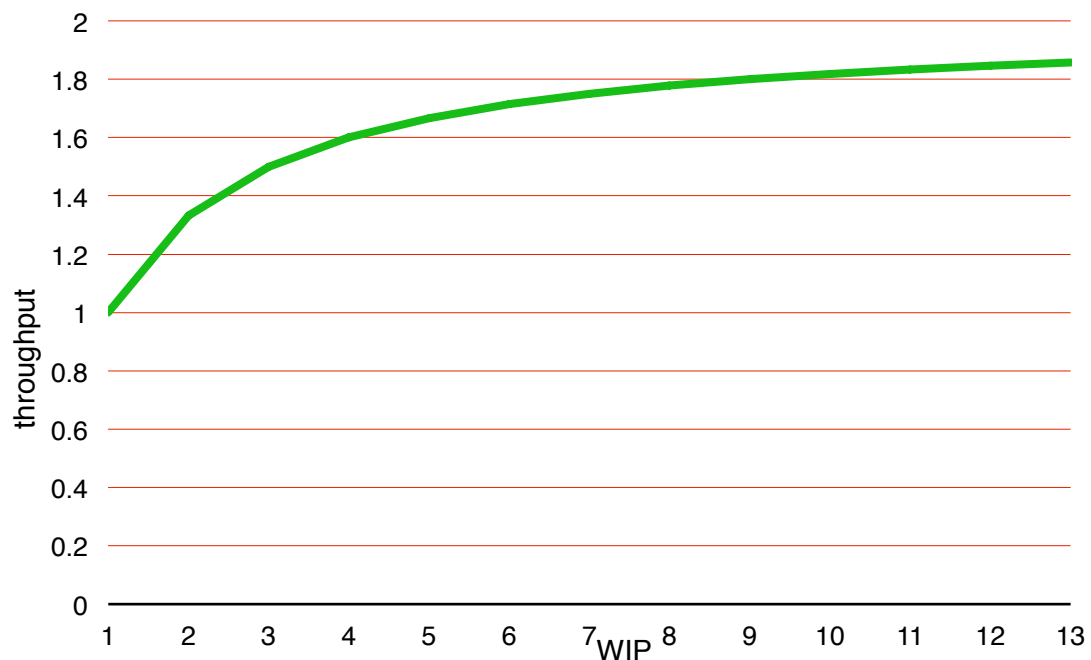
$$WIP_0 = r_b \times T_0 = 2$$

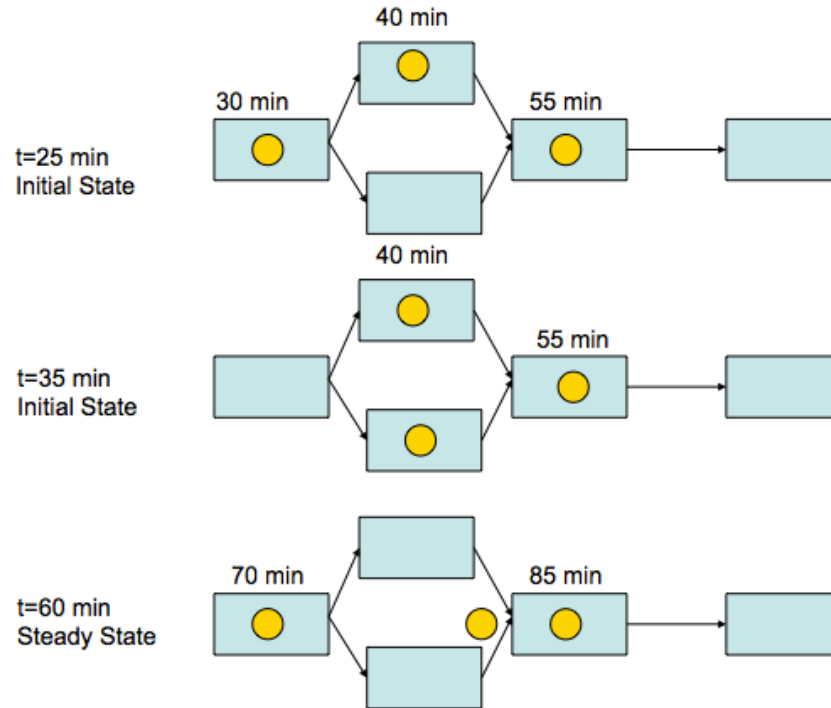
- c) $TH_{PWC} = \frac{WIP}{CT_{PWC}} = \frac{w}{W_0 + w - 1} \cdot r_b$ (VI.10)

$$TH_{PWC} = 3 / (2 + 3 - 1) \times 2 = 6/4 = 1,5 \text{ products per hour}$$

WIP isn't always maximized because the greater WIP becomes, the smaller its influence on performance will be. Furthermore, WIP also means cost in real life.

- d) Steady state
Important measure like throughput are not stable yet. E.g. $TH < TH_{\text{steady_state}}$





- e) $TH_{\text{worst}} = 1 / 1 = 1$ product per hour. When a production line is unable to meet this performance, there is likely to be machine failure or other factors increasing the raw production time T_0 .
- f) In a supply chain, the control over the ordering (KANBAN in a production line) is not centrally organized. The load balancing over the links a supply chain is therefore less organized, leading to oscillating order behavior and stock-outs.
- g) Overdamped system is desirable, because a weakly damped system will permit oscillations. (bullwhip effect).

h)
$$\gamma = \frac{\xi + \delta}{2T}$$

$$\omega_0 = \sqrt{\frac{1}{T\tau}}$$

$$\frac{F_0}{m} \cos(\omega t) = \frac{1}{T} \left(\frac{q_{i+1}(t)}{\tau} + \xi \frac{dq_{i+1}}{dt} \right)$$