

Computational Solid Mechanics /

WMME028-05

Homework

Exercise 1. (1.5 Point) Consider the linear elastic bar in Fig. 1, with length L and constant Young's modulus E , subject to the body force $b(x) = cx$, with c constant, and boundary conditions $u(0) = 0$ and $Eu'(L) = 0$.

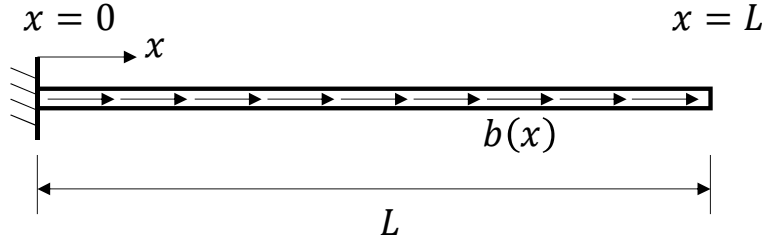


Figure 1: **1D bar with distributed load.** The coordinates and the boundary conditions are also specified.

1. What is the equilibrium equation in strong form (SE)?
2. What are the essential and the natural boundary conditions?
3. What is the exact solution $u(x)$?
4. Starting from (SE), derive the weak form (WE).

Exercise 2. (2 Points) Consider the same bar as in Exercise 1. Do not discretize the bar, but assume a trial function that is defined over the whole domain:

$$u^h(x) = a_0 + a_1x \quad x \in [0, L] \quad (1)$$

In Galerkin's method, the test function $v^h(x)$ has the same degree of approximation as the trial function:

$$v^h(x) = b_0 + b_1x \quad x \in [0, L] \quad (2)$$

1. Determine $u^h(x)$ and $v^h(x)$ after imposing the essential boundary conditions.
2. Substitute $u^h(x)$ and $v^h(x)$ in the weak form, and compute the integrals: what are the parameters of the trial function? [Hint: recall that the (admissible) parameters of the test function are arbitrary]
3. Draw the approximate solution $u^h(x)$ and the exact solution $u(x)$ (from previous Exercise) on the same plot, as a function of the bar coordinate $x \in [0, L]$. What is the maximum absolute error $\epsilon_u = |u^h(x) - u(x)|$ in the displacement by using $u^h(x)$ instead of the exact solution $u(x)$?
4. Draw on the same plot the stress that results from $u^h(x)$ and from the exact solution, as a function of the coordinate $x \in [0, L]$. What is the maximum absolute error in the stress $\epsilon_\sigma = |\sigma^h(x) - \sigma(x)|$ by using the approximate instead of the exact solution?

Exercise 3. (2.5 Points) Consider the 1D quadratic isoparametric 3-nodes element. The mapping $x(\xi)$ between master and actual element is

$$x(\xi) = \frac{(\xi - 1)\xi}{2}x_i + (1 - \xi^2)x_j + \frac{(1 + \xi)\xi}{2}x_k . \quad (3)$$

1. Determine the gradient of the shape functions with respect to the master element coordinates;
2. Determine the Jacobian matrix and the determinant of the change of coordinates, and compute them for the case $(x_i = 0, x_j = 2, x_k = 4)$;
3. Use Gauss integration to compute the element stiffness matrix for the case $(x_i = 0, x_j = 2, x_k = 4)$ and assuming linear elasticity with constant Young's modulus E , considering the following cases:
 - One Gauss point;
 - Two Gauss points.

Does the number of Gauss integration points change the element stiffness matrix? Is this result expected?

4. Further, compute analytically the element stiffness matrix: how do the analytical and the two Gauss integration schemes compare? Is this result expected?
5. For both Gauss integration schemes, assemble and compute the global stiffness matrix, by considering the following connectivity matrix, and assuming that the Young's modulus of element 2 is equal to twice the Young's modulus E of elements 1 and 3:

$$\left[\begin{array}{c|ccc} \text{Element } (m) & \text{Node i } (x_i) & \text{Node j } (x_j) & \text{Node k } (x_k) \\ \hline 1 & 1 \ (0) & 5 \ (2) & 2 \ (4) \\ 2 & 2 \ (4) & 6 \ (6) & 3 \ (8) \\ 3 & 3 \ (8) & 7 \ (10) & 4 \ (12) \end{array} \right] \quad (4)$$

Explain how you perform the assembly (that is, which elements contribute to which slots of the global stiffness matrix). Does the choice of integration scheme affect the entries of the global stiffness matrix?

Exercise 4. (2.5 Points) Consider the deformation described by $\mathbf{F} = \lambda \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{1}{\sqrt{\lambda}} \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3$, with $\mathbf{e}_i$ $i = 1, 2, 3$ defining an orthonormal vector basis and λ a stretch parameter.

1. Determine the relative volume change for this deformation;
2. Consider the hyperelastic constitutive law

$$\boldsymbol{\tau} = \kappa(J - 1)\mathbf{I} + G(\mathbf{B} - \mathbf{I}) \quad (5)$$

where $J = \det(\mathbf{F})$, $\mathbf{I}$ is the 2^{nd} order unit tensor, $\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T$ is the Finger tensor, κ and G are the elastic bulk and shear moduli, respectively.

Write down the matrix representation, in the basis defined by $\mathbf{e}_i$, of the constitutive law in terms of the first Piola-Kirchhoff stress;

3. Compute two non-zero components of your choice of the analytical consistent tangent stiffness $\mathbb{A} := \left(\frac{\partial \mathbf{P}}{\partial \mathbf{F}} \right)$, evaluated at the stretch λ_i ;
4. Consider an isoparametric 2D Quad4 element (Figure 2)

Evaluate, at one Gauss integration point of your choice, the contribution to a non-zero component of the element stiffness matrix $[\mathbf{K}^i]_m$.

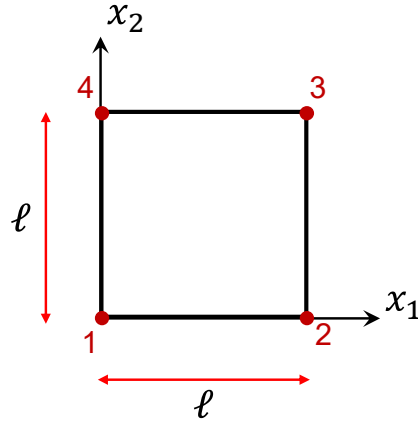


Figure 2: **Isoparametric 2D Quad4 element.**

Exercise 5. (1.5 Point) Consider the 2D finite element discretization (red lines) of the body delimited by the black line in Figure 3. The body is discretized with bilinear elements, that have 4 nodes (one per corner).

1. Write down the element numbers and node numbers. What is the connectivity matrix?
2. What is the total number of degrees of freedom of the FE discretization?
3. What are the dimensions of the element and the global stiffness matrices?
4. Consider element number 3 in your discretization. What are the global degrees of freedom connected to this element? To which slots of the global stiffness matrix does the element stiffness $[\mathbf{K}_3]$ contribute?
5. Which slots of the global stiffness matrix will be non-zero?

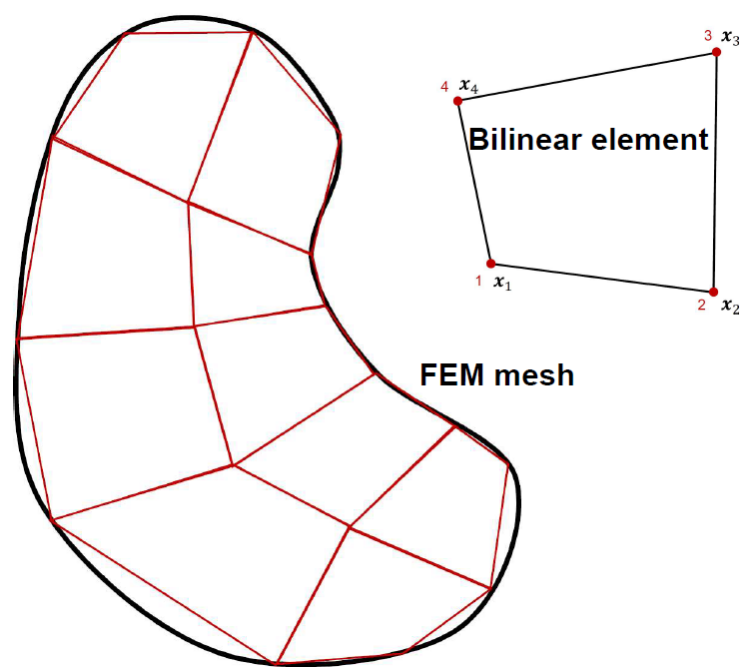


Figure 3: 2D FEM mesh and bilinear element.