

Re-Examination of Physical Systems

May 12, 2015, 18:30-21:30

INSTRUCTIONS (READ THIS CAREFULLY)

1. Make sure you solve each problem on a separate sheet. Solutions of different problems on the same sheet will NOT be looked at.
2. **Write your name and student number clearly on top of every sheet.**
3. Solutions which are not on the right sheet will not be graded. If you need extra paper, use the sheets provided and indicate for which problem the extra sheet is used for.
4. Solve the problems in a systematic way and check your answers. If you think you have made a mistake in a calculation, indicate this. Argue then how you intended to get the right answer and write this down.
5. This is a closed-book examination. No books, notes or graphical calculators may be used during the examination.
6. **Write in a legible manner. Unreadable text will not be handled during grading.**

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Problem #1. Complex Numbers and Differential Equations (9 Points)

- i) Consider the differential equation

$$y'' + \alpha y' + \beta y = 0 \quad (1)$$

Show for which values of α and β the solution takes the form (a and b are either the solutions of the characteristic equation or obtained from them):

- (a) $C_1 e^{at} + C_2 e^{bt}$ **(1.5 Points)**
 - (b) $(C_1 + C_2 t) e^{at}$ **(1.5 Points)**
 - (c) $C_1 e^{at} \cos(bt) + C_2 e^{at} \sin(bt)$ **(1.5 Points)**
- ii) Consider the complex numbers $A = 3 + 4i$, $B = 2 - 3i$, $C = -2 + 4i$ and $D = 1 + 3i$
- (a) Calculate $X = A \cdot B$, $Y = C/D^*$ and $Z = (A^* - B)/(C - D^*)$ and write them in the form $\alpha + \beta i$ **(1.5 Points)**
 - (b) Convert X , Y and Z into Euler notation (with a modulus and a phase) **(1.5 Points)**
 - (c) Draw X , Y and Z in the complex plane **(1.5 Points)**

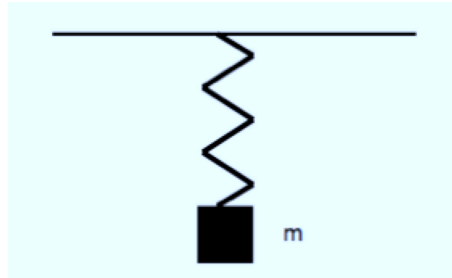
Problem #2. Mechanical Oscillations without Damping (17 Points)

Consider a (mathematical) pendulum. The pendulum consists of a massless, rigid bar with a length $L = 0.75$ m, and a mass, m , of 2 kg hanging at the end. We assume there is no damping.

- a) Make a drawing, draw the gravitational force acting on the pendulum and calculate its longitudinal and tangential components when the pendulum is away from its equilibrium position by 8° . **(4 Points)**
- b) Derive the differential equation of motion for the pendulum. Use $\sin \theta \approx \theta$. **(3 Points)** Under which condition does this approximation apply? **(1 Point)**
- c) Calculate the angular frequency of the system. **(1 Point)** Also, calculate the oscillation time. **(1 Point)**
- d) Show that $\theta = \theta_0 \sin(\omega t + \pi/2)$ is a solution to the equation of motion derived in b). **(3 Points)**
- e) The maximum speed of the pendulum is 0.6 m/s. Calculate the total energy of the system. **(2 Points)**
- f) Calculate the amplitude θ_0 of the oscillation. **(2 Points)**

(Note: $d\theta/dt$ does not correspond to the speed in m/s.)

Problem #3. Mechanical Oscillations with Damping (18 Points)



Assume a spring system as above with a mass of 1 kg at the end of it. The top point of the spring is driven with an external force $F_0 = 0.08$ N. The spring constant is 4 kg/s² and the damping factor 0.1 s⁻¹. The full differential equation for an externally driven, damped oscillation is given by:

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2x = \frac{F_0}{m}\cos(\omega t)$$

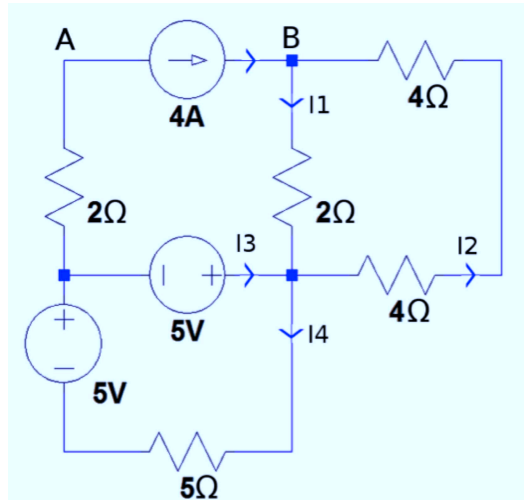
The full solution for this is:

$$x(t) = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + (2\omega\gamma)^2}}\cos(\omega t) + Ae^{-\gamma t}\cos(\sqrt{\omega_0^2 - \gamma^2}t)$$

- The full solution consists of two parts. Using this solution, write down the stationary solution of the system and the transient solution of the system, separately **(2 Points)** and explain what they physically mean **(2 Points)**. Is this a weakly, critically or over-damped oscillation? why? **(2 Points)**
- Derive the expression for the acceleration of the transient solution at $t = 0$. **(3 Points)**
- Calculate the value of the stationary amplitude for an angular frequency ω of 3 rad/s. **(2 Points)**
-) For $\omega = \omega_0$, the stationary amplitude will be at its largest. How do you call this phenomena? **(1 Point)** and how big is this amplitude? **(1 Point)**
- Write down the expression of the total energy of the stationary motion as a function of ω . (Hint: You can use the maximum potential energy). Show the steps how you get there. **(3 Points)** How big is this energy at the resonance frequency? **(2 Points)**

Problem #4. Electric Circuit – I (17 Points)

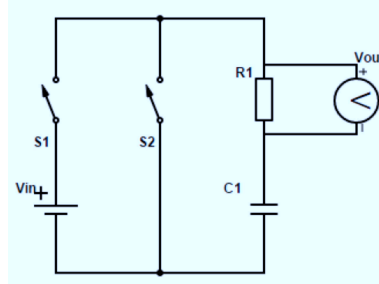
Consider the electrical circuit below:



- Write down the first and the second laws of Kirchhoff (**2 Points**) and explain what they physically mean. (**2 Points**)
- Find all the node equations (Kirchhoff 1) and write them down. (**3 Points**)
Show that one of these equations can be derived from the others. (**1 Points**)
- Find all the loop equations (Kirchhoff 2) and write them down. (**4 Points**)
- Find I_1 , I_2 , I_3 , and I_4 and the potential difference $V_{AB} = V_B - V_A$ over the current source. (**Total of 5 Points, 1 Point each**)

Problem #5. Electric Circuit – II (19 Points)

In the following electrical circuit, $V_{in} = 100 \text{ V}$, $R1 = 300 \Omega$ and $C1 = 5 \mu\text{F}$.



Initially, there is no current running through the circuit. At time $t = 0$, switch $S1$ closes and current starts to flow through the circuit.

- Show that the current from that moment on ($t > 0$) is given by: **(4 Points)**
 $I(t) = \frac{V_{in}}{R} e^{-t/RC}$. (HINT: use Kirchhoff's law for voltage over the loop and proceed with your derivation.)
- What is the characteristic time for this RC circuit? **(2 Points)** Show that this unit is indeed seconds. **(1 Point)**

After a certain amount of time, the voltage V becomes zero and the voltage over $C1$, V_C reaches the value of the power supply, V_{in} . Then switch $S1$ is opened and $S2$ is closed causing the circuit to discharge. Hence, the effect of the voltage supply is taken away.

- How long will it take until the voltage over the capacitor is only 20% of the value it had at the moment that the switches $S1$ and $S2$ were switched. (HINT: when discharging a fully-charged capacitor, the following formula holds: $V_C(t) = V_{in} e^{-t/RC}$). **(2 Points)**

We now change the DC voltage source to an AC voltage source with frequency ω .

- Show that the voltage transfer $|V_{out}/V_{in}|$ is then given by: **(4 Points)**

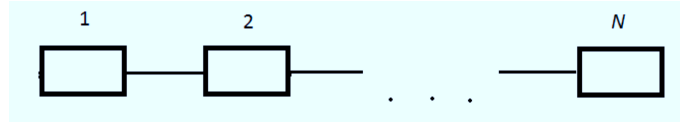
$$\frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}$$

- Calculate the cut-off frequency of this system. (HINT: Remember that the cut-off frequency is the frequency at which the above ratio is $1/\sqrt{2}$.) **(2 Points)**
- Calculate the phase difference ϕ between V_{in} and V_{out} at this cut-off frequency? **(2 Points)**
- The above circuit will act as a filter. What is such a filter generally called? **(1 Point)** Draw a graph of the transfer as function of ω to support your reasoning. **(1 Point)**

Problem #6. Business Dynamics (20 Points)

- a) Give the definition of the utilization of a workstation. **(3 Points)**

Consider a system with a sequential production line of N one machine work stations (see below):



There is a work-in-progress level of w and each machine has an average production time t .

- b) Give the raw processing time (T_0) and the bottleneck rate (r_b) of this production line. **(2 Points)** What is the definition of the bottleneck rate? **(1 Point)** How can the bottleneck rate of the production line above be improved if it is not allowed to have more than one machine per workstation? **(2 Points)**

The cycle time of one machine ($CT_{machine}$) is given by the processing time for one job plus the average time it takes for other jobs to be finished before that one job can be processed ($t + \frac{(w-1)}{N}t$).

- c) Use this equation to derive the equations for the practical worst case cycle time (CTPWC) and the practical worst case throughput (THPWC) for this production line in terms of w , r_b and critical WIP (W_0). **(4 Points)**

The article by Lee et al. (1997) describes the bullwhip effect in supply chains, the major causes for this and ways to counteract this effect.

- d) Name three ways to counteract the bullwhip effect and briefly describe them. **(3 Points)**

The supply chain oscillation can be described by the following equation:

$$\frac{d^2 q_i(t)}{dt^2} + \frac{(\beta + \epsilon)}{T} \frac{dq_i(t)}{dt} + \frac{1}{T \cdot \tau} q_i(t) = \frac{1}{T} \left(\frac{q_{i+1}(t)}{\tau} + \beta \frac{dq_{i+1}(t)}{dt} \right)$$

- e) What does the term on the right hand side represent? What does the $q_{i+1}(t)$ represent in this term? **(2 Points)**

Just like in mechanical and electrical systems, three kinds of oscillation can occur; a weakly damped oscillation, an over-damped oscillation and critical damping.

- f) Give the (mathematical) conditions for all three oscillations in case of the business system. **(3 Points)**

1) Complex Numbers and Differential Equation

i)

(a) $\alpha^2 > 4\beta$

(b) $\alpha^2 = 4\beta$

(c) $\alpha^2 < 4\beta$

ii)

(a) $18 - i$, $-1.4 - 0.2i$ and $-5/29 - 2i/29$. For correct method and wrong answers, partial points are given.

(b) $18e^{-3.18i}$, $1.41e^{-172i}$ and $0.186e^{-158.2i}$ (angles expressed in radians are also OK. Again for correct method and depending on your answers in A, partial points are given.)

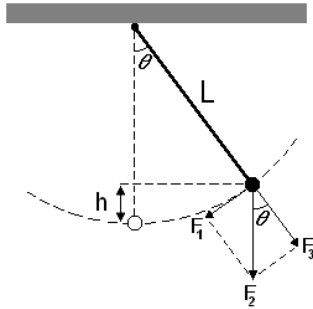
(c) use part (a) and the plot the numbers in the imaginary vs. real axis.

2) Mechanical systems without damping

a) $F_z = F_2 = m g = 2 * 9.81 = 19.62 \text{ N}$

$\sin(8^\circ) = F_1/F_2 \rightarrow F_T = F_1 = \sin(8^\circ) * 19.62 = 2.73 \text{ N}$

$\cos(8^\circ) = F_3/F_2 \rightarrow F_L = F_3 = \cos(8^\circ) * 19.62 = 19.43 \text{ N}$



b) $\frac{d^2\theta}{dt^2} + \frac{g}{L} \theta = 0$

This approximation applies when θ has a sufficient low value.

c) $\omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.81}{0.75}} = 3.62 \text{ rad/s}$

$T = 2\pi/\omega = 1.737 \text{ s}$

d) $\theta = \theta_0 \sin(\omega t + \pi/2)$

$\frac{d\theta}{dt} = \omega \theta_0 \cos(\omega t + \pi/2)$

$\frac{d^2\theta}{dt^2} = -\omega^2 \theta_0 \sin(\omega t + \pi/2)$

$\frac{d^2\theta}{dt^2} + \frac{g}{L} \theta = \frac{d^2\theta}{dt^2} + \omega^2 \theta = 0$

$-\omega^2 \theta_0 \sin(\omega t + \pi/2) + \omega^2 \theta_0 \sin(\omega t + \pi/2) = 0 \rightarrow \text{correct}$

e) $E_{tot} = E_{kin,max}$

$E_{kin,max} = \frac{1}{2} m v_{max}^2 = \frac{1}{2} * 2 * 0.6^2 = 0.36 \text{ J}$

f) $E_{kin,max} = E_{pot,max}$

$m g h = 2 * 9.81 * h = 0.36 \rightarrow h = 0.0183 \text{ m}$

$L - h = 0.75 - 0.0183 = 0.732 \text{ m}$

$\cos\theta_0 = (L-h) / L = 0.732/0.75 \rightarrow \theta_0 = 12.6^\circ$

3) Mechanical systems with damping

a) Stationary solution: $x_{stat}(t) = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + (2\omega\gamma)^2}} \cos(\omega t)$

Transient solution: $x_{trans}(t) = Ae^{-\gamma t} \cos(\sqrt{\omega_0^2 - \gamma^2} t)$

The oscillation is weakly damped as $\omega_0^2 - \gamma^2 > 0$ ($\omega_0 = 2 \text{ s}^{-1}$ and $\gamma = 0.1 \text{ s}^{-1}$)

b) $x(t) = Ae^{-\gamma t} \cos(\omega t)$

$v(t) = x'(t) = -\gamma Ae^{-\gamma t} \cos(\omega t) - \omega Ae^{-\gamma t} \sin(\omega t)$

$a(t) = x''(t) = \gamma^2 Ae^{-\gamma t} \cos(\omega t) + \omega\gamma Ae^{-\gamma t} \sin(\omega t) + \omega\gamma Ae^{-\gamma t} \sin(\omega t) - \omega^2 Ae^{-\gamma t} \cos(\omega t)$

$a(t) = \cos(\omega t) [\gamma^2 Ae^{-\gamma t} - \omega^2 Ae^{-\gamma t}] + \sin(\omega t) [\omega\gamma Ae^{-\gamma t} + \omega\gamma Ae^{-\gamma t}]$

for $t = 0$: $a(0) = \gamma^2 Ae^{-\gamma \cdot 0} - \omega^2 Ae^{-\gamma \cdot 0} = \gamma^2 A - \omega^2 A = A(\gamma^2 - \omega^2)$

c) $A_{stat} = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + (2\omega\gamma)^2}}$

With $F_0 = 0.08 \text{ N}$, $m = 1 \text{ kg}$, $\gamma = 0.1 \text{ s}^{-1}$, ω of 3 rad/s and $k = 4 \text{ kg/s}^2 \rightarrow \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{1}} = 2 \text{ rad/s}$

$A_{stat} = \frac{0.08/1}{\sqrt{(3^2 - 2^2)^2 + (2 \cdot 3 \cdot 0.1)^2}} = 0.0159 \text{ m}$

d) Resonance for which $\omega = \omega_0 = 2 \text{ rad/s}$

$A_{stat} = \frac{0.08/1}{\sqrt{(0)^2 + (2 \cdot 2 \cdot 0.1)^2}} = 0.2 \text{ m}$

e) $E_{tot} = E_{pot} + E_{kin} = \frac{1}{2} k x^2 + \frac{1}{2} m v^2$

For maximum displacement: $E_{tot} = E_{pot} = \frac{1}{2} k x^2 = \frac{1}{2} k A^2$

$E_{tot} = \frac{1}{2} k \left[\frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + (2\omega\gamma)^2}} \right]^2$

For $\omega = \omega_0$, $A = 0.2 \text{ m}$

$E_{tot} = \frac{1}{2} k A^2 = \frac{1}{2} \cdot 4 \cdot 0.2^2 = 0.08 \text{ J}$

4) Electrical Circuits I

a) First law $I_1 + I_2 - I_0 = 0$ or the sum of current is zero taking the directions into account. the “sum” of the currents at any node is zero.

Second law: $V - V_{r1} - V_{r2} = 0$, the “sum” of the voltages in any circuit loop equals zero.

b) Using the first law

$$(1): 4 - I_1 + I_2 = 0$$

$$(2): I_1 + I_3 - I_2 - I_4 = 0$$

$$(3): I_4 - I_3 - 4 = 0$$

$$-((1)+(2)) = (3)$$

c) Leaving out the current source we have:

$$(4): 2I_1 + 4I_2 + 4I_2 = 0$$

$$(5): 5 + 5 - 5I_4 = 0$$

d) From (5): $I_4 = 2A \rightarrow (3): I_3 = -2A \rightarrow (4): I_1 = -4I_2$ combine with (1) and find: $I_2 = I_1 - 4 = -4I_2 - 4$ and $I_2 = -4/5 A$. Use (1): $I_1 = 16/5 A$

The potential difference over the current source results from Kirchhoff 2 for the loop with the current source (clockwise):

$$-8 + V_{ab} - 2I_1 - 5 = 0 \rightarrow V_{ab} = 13 + 2I_1 = 13 + 32/5 = 19 \frac{2}{5} V$$

5) Electrical Circuits II

a) See reader p. 36/37. When the circuit closes, the sum of the voltages across the resistor R1 and capacitor C1 will be equal to the voltage of the voltage supply V_{in} . So:

$$V_{in} = I \cdot R + \frac{Q}{C}$$

Differentiate both sides in time results in: $0 = R \frac{dI}{dt} + \frac{1}{C} \frac{dQ}{dt}$

with dQ/dt always equal to I so we write: $0 = RC \frac{dI}{dt} + I$

and $\frac{dI}{dt} = -\frac{1}{RC} I$

This simple differential equation results in $I(t) = K \cdot e^{-\frac{t}{RC}}$

At $t = 0$ the current is determined by the resistor to be $\frac{V_{in}}{R}$, so $K = \frac{V_{in}}{R}$.

And we indeed get $I(t) = \frac{V_{in}}{R} \cdot e^{-\frac{t}{RC}}$

b) This time is also known as the RC time (p. 37), so is determined by the product of R1 and C1 in this case, giving 0.0015 s or 1.5 ms.

The unit is: $R = \Omega = \frac{\text{Volt}}{\text{Ampere}}$ and $C = F = \frac{\text{Coulomb}}{\text{Volt}} = \frac{\text{Ampere} \cdot \text{second}}{\text{Volt}}$ so we get

$\frac{\text{Volt} \cdot \text{Ampere} \cdot \text{second}}{\text{Ampere} \cdot \text{Volt}}$ which results in seconds [s]. So 0.0015 seconds or 1.5 ms.

c) Only possible if e-power in the formula is equal to 0.2:

$$V_C(t) = V_{in} \left(e^{-\frac{t}{RC}} \right)$$

So, $e^{-\frac{t}{RC}} = 0.2 \rightarrow -\frac{t}{RC} = \ln(0.2) \rightarrow t = RC \cdot \ln(0.2^{-1}) = 0.0024 \text{ seconds}$

d) See reader p. 46. V_R is V_{out} so the transfer is $\frac{Z_R}{Z_C + Z_R}$.

We get: $\frac{R}{-\frac{i}{\omega C} + R}$

Multiply by complex conjugate of denominator to get:

$$\frac{R^2 + \frac{iR}{\omega C}}{R^2 + \frac{1}{(\omega C)^2}}$$

Multiply now by $\omega^2 C^2$:

$$A_R = \frac{(\omega RC)^2 + i\omega RC}{1 + (\omega RC)^2}$$

Voltage transfer is given as the modulus of the transfer, so calculate $|A_R|$:

$$\begin{aligned} |A_R| &= \sqrt{\frac{(\omega RC)^4 + (\omega RC)^2}{(1 + (\omega RC)^2)^2}} \\ &= \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} \end{aligned}$$

e) Cut-off frequency occurs when $\omega = \frac{1}{RC}$

Filling in results in $\omega = 666.67 \text{ rad/s}$.

f) Phase difference φ is given by

$$\tan \varphi = \frac{\text{Im}(A_R)}{\text{Re}(A_R)}$$

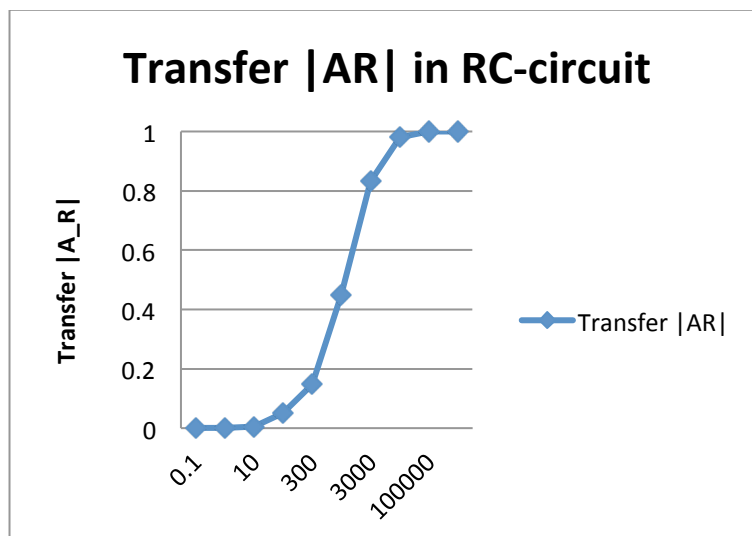
Which gives here:

$$\tan \varphi = \frac{\frac{1}{\omega C}}{R} = 0.78539 \text{ rad}$$

Which is equivalent to $\pi/4$.

g) Above should ring enough bells that we deal with a high-frequency RC filter.

A_R graph should have low transfer for low ω and high transfer for high ω (or plug in $\omega \rightarrow 0$ and $\omega \rightarrow \infty$) to obtain:



6) Business Dynamics

a) The definition of the utilization of a workstation is the fraction of time it is not idle for lack of parts (the fraction of time a workstation is in use). The utilization includes machine failure or set-up times.

b) $T_0 = N \cdot t$, $r_b = 1/t$. The bottleneck rate is the rate of the workstation having the highest utilization. The bottleneck rate can be improved by decreasing the average production time t .

c)

$$CT_{PWC} = N \cdot t + (w-1)t =$$

$$T_0 + (w-1)/r_b =$$

$$W_0/r_b + (w-1)/r_b,$$

$$TH_{PWC} =$$

$$WIP/CT_{PWC} =$$

$$w/(W_0/r_b + (w-1)/r_b) =$$

$$(w \cdot r_b)/(W_0 + (w-1)).$$

d) Avoid multiple demand forecast updates, by getting access to downstream demand information due to data sharing or bypassing downstream sites, multiple demand forecasts can be decreased and unnecessary inventory fluctuations with it. Break order batches, smaller batches or more frequent resupply lead to smaller inventory fluctuations. Stabilize prices, decreases the bullwhip effect by reducing forward buying and diversion. Eliminate gaming in shortage situations, decrease the exaggerated orders by customers when supply is low and the sudden drop in demand when supply is high again.

e) The RHS represents the driving factor. The q_{i+1} represents the change in delivery rate to the customer (in response to the demand).

f) Weakly damped: $0 < (\beta + \epsilon) < 2\sqrt{T/\tau}$. Over-damped: $(\beta + \epsilon) > 2\sqrt{T/\tau}$. Critical damping: $(\beta + \epsilon) = 2\sqrt{T/\tau}$.