

Final Examination of Physical Systems

April 10, 2015, 8:30-11:30

INSTRUCTIONS (READ THIS CAREFULLY)

1. **Make sure you solve each problem on a separate sheet.**
2. Write your name and student number clearly on top of every sheet.
3. Solutions which are not on the right sheet will not be graded. If you need extra paper, use the sheets provided and indicate for which problem the extra sheet is used for.
4. Solve the problems in a systematic way and check your answers. If you think you have made a mistake in a calculation, indicate this. Argue then how you intended to get the right answer and write this down.
5. This is a closed-book examination. No books, notes or graphical calculators may be used during the examination.
6. **Write in a legible manner. Unreadable text will not be handled during grading.**

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Problem #1. Complex Numbers and Differential Equations (10 Points)

1. Consider the differential equation

$$x' = (a + t)x^2t \quad (1)$$

- a) Solve the differential equation. **(3 Points)**
 - b) Give the values for a and C_1 which give a solution to the boundary value problem with $x(0) = 2$ and $x(1) = 1$. **(2 Points)**
2. Consider the complex function

$$x(t) = e^{-t}e^{i\omega t} \quad (2)$$

- a) Sketch the real part of $x(t)$ for $t > 0$ for a given value of ω . **(2 Points)**
- b) At what values of t do maxima, minima and roots (zeros of x) occur for the real part? Explain this with a drawing in the complex plane. **(2 Points)**
- c) What is the name of such a function which represents a physical system? **(1 Point)**

Problem #2. Mechanical Oscillations without Damping (17 Points)

A mass of m kg is horizontally connected to a spring with a spring constant k N/m and undergoes a harmonic vibration with an amplitude A . Assume there is no friction.

- a) What is the force balance of this system? Use this balance to give the differential equation that is appropriate for the motion. **(3 Points)**

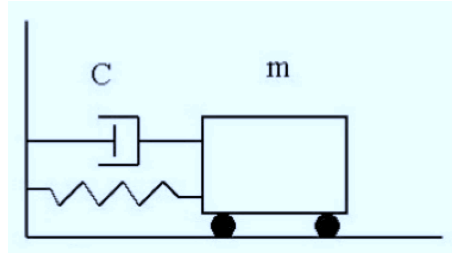
At $t = 0$, the mass is released at its maximum position.

- b) Give a solution for the differential equation of the motion. You can simply guess a solution and make sure it is the right one. **(3 Points)**

Now it is given that $m = 1.5$ kg, $k = 24$ N/m and the maximum displacement is 40 cm.

- c) Calculate the maximum acceleration of the mass. In which position(s) does this take place? **(3 Points)**
- d) In which position x is the mass when the speed is -1.5 m/s? **(3 Points)**
- e) Calculate the total energy of the system. **(3 Points)**
- f) We take another identical spring and connect it to the first one so that they form a long spring (in series) which is connected to the mass. What is the new value for the angular frequency? And for the total energy? **(2 Points)**

Problem #3. Mechanical Oscillations with Damping (17 Points)



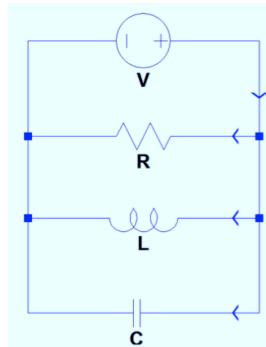
Mass m is horizontally connected to a spring with a spring constant k in the way that is shown in the sketch above. The strength of the damping is represented by C which leads to a friction force of $F_{\text{friction}} = -Cdx/dt$.

- Write the equation that contains all the forces acting on this system. **(4 Points)**
- Give the general equation of motion of this damped oscillation in the form of a differential equation and give the expression for the eigen frequency ω_0 and replace the coefficient of dx/dt by 2γ . **(3 Points)**
- Show that $x(t) = Be^{-\gamma t} \cos(\omega t)$ is a solution of this equation and give an expression of ω as a function of ω_0 and γ . **(4 Points)**

The frequency f of this system is 4 Hz and the damping factor γ is 0.2 s^{-1} .

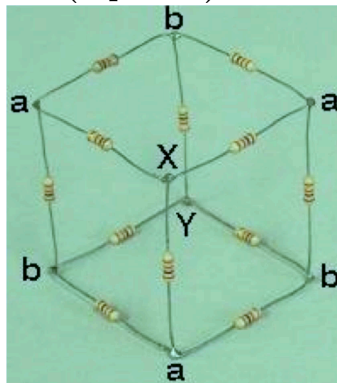
- What is the percentage of the original amplitude that is left after 6 oscillations? **(3 Points)**
- The solution $x(t)$ here is correct for weak damping. What are the other forms of damping that can occur? What are the values of γ compared to ω_0 in these forms? Draw a graph for all three situations. **(3 Points)**

Problem #4. Electric Circuit – I (18 Points)



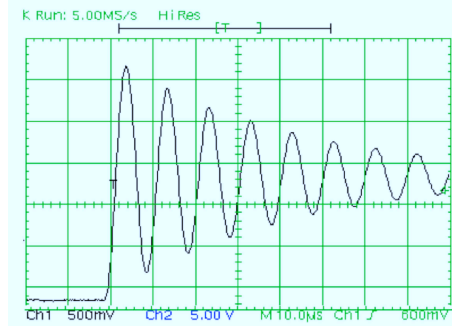
Above is an RLC circuit, but instead of the usual circuit represented in series, this one is parallel.

- Write down the first and second laws of Kirchhoff and explain what they mean. **(4 points)**
- Write down the four equations for the above circuit which you can derive using the laws of Kirchhoff. **(4 points)**
- Calculate I_R , I_L and I_C in terms of V , R , L , C and t . **(4 points)**
- Generally, one has to look at these currents individually (as they are generally different) and solve the problem further. Now, and for the sake of simplicity, suppose I_R , I_L and I_C all have the same value, namely I . Write down the differential equation for I and its solution. (Hint: Use the equations from the laws of Kirchhoff.) **(4 points)**
- Using simple symmetry arguments (by following the current which goes from one corner, X, to the other corner, Y), reduce the following three dimensional structure into a two-dimensional one and calculate the equivalent resistance between X and Y. All the resistors in the picture have the same resistance R . Note that you do NOT have to write all the equations by using the Kirchhoff's laws to solve this problem. **(2 points)**



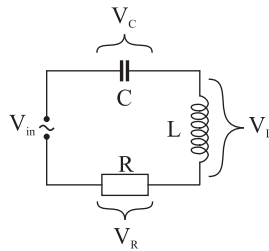
Problem #5. Electric Circuit – II (18 Points)

Given is the following output (connected to Ch2) of an oscilloscope:



- a) Determine the eigen-frequency ω_0 of the system that generates this signal. Hint: just read off the numbers from the picture. The time scale is $10 \mu\text{s}$ per division. **(3 Points)**

Assume this signal is produced by the following LCR-circuit (at resonance, the frequency is $1/\sqrt{LC}$):



- b) What inductance L is used to give the system this eigen-frequency ω_0 if the capacitance C is $1.5 \mu\text{F}$. (if no value of ω_0 was found in part a), use $5 \times 10^5 \text{ rad/s}$). **(3 Points)**
- c) As in the mechanical domain, we have three damping regimes. Which one do you recognize in the oscillation shown in the first figure? Explain. **(2 Points)**
- d) Calculate the value of the resistance R needed to critically damp this system (Hint: $2\gamma = R/L$). What would the oscillation look like if it becomes critically damped? Draw this. **(3 Points)**
- e) Write down the expression for the general total (complex) impedance of an LCR-circuit. **(2 Points)**
- f) Show that the voltage transfer $|A|$ over the inductor is equal to $\frac{\omega L}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$. **(3 Points)**

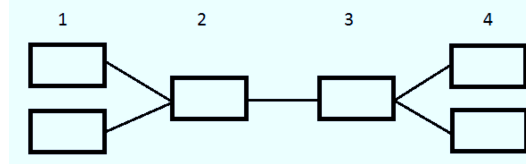
Besides the measurement signal, there is an disturbance signal of 15 kHz .

- g) What is the transfer coefficient for this disturbance signal? Express this ratio in terms of deciBel (dB). **(2 Points)**

Problem #6. Business Dynamics (20 Points)

- a) The business equivalent of Ohms law is known as Little's law and is given by $WIP = TH \times CT$. Give the definitions of all of the components of Little's Law. Include the dimension in which each component is specified (e.g. speed: [meters/second]). **(3 Points)**

Consider the production line with 4 stations below (products move from station 1 to 4).



The processing times of each station are 8, 5, 2 and 6 hours for stations 1, 2, 3 and 4, respectively. Consider one piece flow.

- b) Calculate the production rate of each process step and indicate which process is the bottleneck process and give the corresponding r_b including its dimension. **(2 Points)**
- c) Calculate the raw production time (T_0) and the critical work-in-progress (W_0) and use these numbers to draw one graph for the cycle time and one graph for the throughput as a function of the WIP in the best case scenario. Indicate clearly where W_0 is placed in both graphs and r_b and T_0 in the corresponding graph. **(4 Points)**

The production line is part of a supply chain. Instabilities have been extensively studied in supply chains. The computer and systems engineer Jay Forrester was the first to draw a comparison to AC currents. The phenomenon he described is known as the bullwhip or Forrester effect.

- d) Explain how the bullwhip/Forrester effect works in supply chains and name three sources that can cause this effect according to the article by Lee et al. (1997). **(4 Points)**

The supply chain oscillation can be described by the following equation:

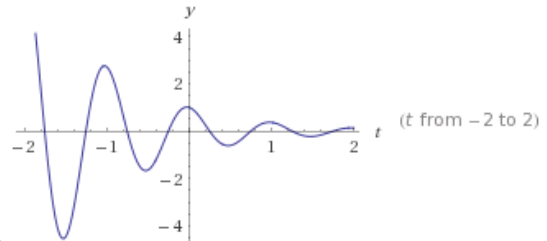
$$\frac{d^2 q_i(t)}{dt^2} + \frac{(\beta + \epsilon)}{T} \frac{dq_i(t)}{dt} + \frac{1}{T \cdot \tau} q_i(t) = \frac{1}{T} \left(\frac{q_{i+1}(t)}{\tau} + \beta \frac{dq_{i+1}(t)}{dt} \right)$$

- e) Derive the expressions for the business dynamics equivalents of the damping factor (γ) and the eigen frequency (ω_0) using the equation above. **(3 Points)**
- f) Use the solutions for the mass-spring (mechanical) system and the expressions for the damping factor and eigen frequency found in e) to derive the solutions for the business system. When does damping occur in these solutions? What boundary condition does this lay upon the supply chain in order to be stable? (HINT: the possible solutions were obtained by solving $\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$.) **(4 Points)**

1) Complex Numbers and Differential Equation

1.a) $-6/(C_1 + (2t + 3a)t^2)$

1.b) $C_1 = -3(x(0))$ thus $a = -5/3 (x(1))$



2.a) Here is the plot for a larger range than asked for.

2.b) $t = n2\pi/\omega$ for maxima, $t = \pi/\omega + n2\pi/\omega$ for minima, $t = \pi/2\omega + n\pi/\omega$.

Draw a spiral starting at the x-axis and spiralling inwards counter-clockwise.

2.c) Weakly damped harmonic oscillation

2) Mechanical systems without damping

a) $m a = -k x$

$$m \frac{d^2 x}{dt^2} + k x = 0$$

b) $x(t) = A \cos(\omega t)$ Or $x(t) = A \sin(\omega t + \pi/2)$

c) $x(t) = A \cos(\omega t)$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{24}{1.5}} = 4 \text{ rad/s}$$

$$v(t) = x'(t) = -\omega A \sin(\omega t)$$

$$a(t) = x''(t) = -\omega^2 A \cos(\omega t)$$

$$a_{max} = \omega^2 A = 4^2 * 0.4 = 6.4 \text{ m/s}^2$$

This takes place at the maximum positions, so at $x = 0.4 \text{ m}$ and $x = -0.4 \text{ m}$

d) $v(t) = -\omega A \sin(\omega t) = -4 * 0.4 \sin(4 t) = -1.5$

$$\sin(4 t) = 0.938, 4 t = 1.215 \text{ and } t = 0.304 \text{ s}$$

$$x(0.304) = 0.4 \cos(4 * 0.304) = 0.139 \text{ m}$$

$$\text{OR } v(t) = \omega A \cos(\omega t + \pi/2) = 4 * 0.4 \cos(4 t + \pi/2) = -1.5$$

$$\cos(4 t + \pi/2) = -0.9375, \text{ then } 4 t + \pi/2 = 2.786. \text{ So, } 4t = 1.215 \text{ and } t = 0.304.$$

$$x(0.304) = 0.4 \sin(4 * 0.304 + \pi/2) = 0.139 \text{ m}$$

e) $E_{tot} = E_{pot} + E_{kin} = \frac{1}{2} k x^2 + \frac{1}{2} m v^2$

$$E_{tot} = E_{pot, max} = \frac{1}{2} k A^2 = \frac{1}{2} * 24 * 0.4^2 = 1.92 \text{ J}$$

$$\text{OR } v_{max} = \omega A = 4 * 0.4 = 1.6 \text{ m/s}$$

$$E_{tot} = E_{kin, max} = \frac{1}{2} m v_{max}^2 = \frac{1}{2} * 1.5 * 1.6^2 = 1.92 \text{ J}$$

f) $k_{eff} = \frac{1}{2} k = 12 \text{ N/m}$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{12}{1.5}} = 2\sqrt{2} \text{ rad/s}$$

$$E_{tot} = E_{pot, max} = \frac{1}{2} k A^2 = \frac{1}{2} * 12 * 0.4^2 = 0.96 \text{ J}$$

$$\text{OR } v_{max} = \omega A = 2\sqrt{2} * 0.4 = 1.13 \text{ m/s}$$

$$E_{tot} = E_{kin, max} = \frac{1}{2} m v_{max}^2 = \frac{1}{2} * 1.5 * 1.13^2 = 0.96 \text{ J}$$

3) Mechanical systems with damping

a) $F = m \cdot a = m \cdot \frac{d^2x}{dt^2}$

$$F_s = -k \cdot x$$

$$F_{friction} = -C \cdot v = -C \cdot \frac{dx}{dt}$$

$$F = F_s + F_{friction}$$

$$m \cdot \frac{d^2x}{dt^2} = -k \cdot x - C \cdot \frac{dx}{dt}$$

b) $\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = 0$

c) $x(t) = Be^{-\gamma t} \cos(\omega t)$

$$x'(t) = -\gamma Be^{-\gamma t} \cos(\omega t) - Be^{-\gamma t} \omega \sin(\omega t)$$

$$x''(t) = \gamma^2 Be^{-\gamma t} \cos(\omega t) + \gamma Be^{-\gamma t} \omega \sin(\omega t) + \gamma Be^{-\gamma t} \omega \sin(\omega t) - Be^{-\gamma t} \omega^2 \cos(\omega t)$$

Put into equation:

$$\gamma^2 Be^{-\gamma t} \cos(\omega t) + \gamma Be^{-\gamma t} \omega \sin(\omega t) + \gamma Be^{-\gamma t} \omega \sin(\omega t) - Be^{-\gamma t} \omega^2 \cos(\omega t) - 2\gamma^2 Be^{-\gamma t} \cos(\omega t) - 2\gamma Be^{-\gamma t} \omega \sin(\omega t) + \omega_0^2 Be^{-\gamma t} \cos(\omega t) = 0$$

$$\sin(\omega t) [\gamma Be^{-\gamma t} \omega + \gamma Be^{-\gamma t} \omega - 2\gamma Be^{-\gamma t} \omega] = \sin(\omega t) \cdot 0 = 0 \rightarrow \text{always}$$

$$\cos(\omega t) [\gamma^2 Be^{-\gamma t} - Be^{-\gamma t} \omega^2 - 2\gamma^2 Be^{-\gamma t} + \omega_0^2 Be^{-\gamma t}] = 0$$

$$\text{In order for this to hold always: } [\gamma^2 Be^{-\gamma t} - Be^{-\gamma t} \omega^2 - 2\gamma^2 Be^{-\gamma t} + \omega_0^2 Be^{-\gamma t}] = 0$$

$$Be^{-\gamma t} [-\gamma^2 - \omega^2 + \omega_0^2] = 0$$

$$\text{In order for this to hold always: } [-\gamma^2 - \omega^2 + \omega_0^2] = 0$$

$$\omega^2 = \omega_0^2 - \gamma^2$$

d) $f = 1/T \rightarrow T = 1/f = \frac{1}{4} = 0.25 \text{ s}$

$$x(t) = Be^{-\gamma t} \cos(\omega t)$$

At $t = 0$, $\cos(\omega t) = 1$. So maximum amplitude of B

After six oscillations, $t = 6 \cdot T = 1.5 \text{ s}$

$$A = Be^{-\gamma t} = Be^{-0.2 \cdot 1.5} = 0.74 B$$

So 74% of the original amplitude is still left

e) Critical damping : $\gamma = \omega_0$

Over-damping: $\gamma > \omega_0$. See figure II.8 of the reader for the graph

4) Electrical Circuits I

a) First law $I_1 + I_2 + (\dots) - I_0 = 0$, the current at a node is zero.

second law $V - V_{r1} - V_{r2} - \dots = 0$, the voltage in a circuit loop equals zero.

b) $V - I_R R = 0$ $I_R R + 1/L * dI_L/dt = 0$ $-1/L * dI_L/dt - Q/C = 0$ $I_0 = I_R + I_L + I_C$

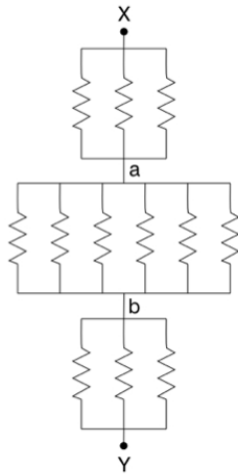
c) $I_R = V/R$ $I_C = C dV/dt$ $I_L = 1/L \int V dt$

These can all be derived from kirchhoff's second law. Also $V = V_R = V_L = V_C$

d) From $IR + 1/L * dI/dt = 0$ $-1/L * dI/dt - Q/C = 0$ follows $IR - Q/C = 0$, differentiate this to get

$dI/dt - I/RC = 0$ $I = I_0 * \exp[-t/RC]$. Note, that this does not represent a real system as the currents will generally not be the same in the three branches.

e)



$$R_{\text{equiv}} = R/3 + R/3 + R/6 = 5R/6$$

5) Electrical Circuits II

a) First you need the frequency in Hz of the signal. The time scale is 10 μ s per division. The time of 1 oscillation takes roughly 1 division, so $T = 10 \mu$ s. We know $T = 1/f$, so $f = 100,000$ Hz. ω_0 is in rad/s so $\omega_0 = 100,000 \cdot 2\pi = 628318.53$ rad/s.

b) $\omega_0 = \frac{1}{\sqrt{LC}}$ so $L = \frac{1}{C\omega_0^2} = 1.7 \cdot 10^{-6} = 1.7 \mu$ H. If using $5.0 \cdot 10^5$ rad/s, you get $L = 2.66 \mu$ H.

c) Many oscillations still occur so it is weakly damped. It approaches the equilibrium point fast, but keeps oscillating around this point, hence weakly damped.

d) $R = 2\sqrt{\frac{L}{C}}$ so filling it in for the calculated values gives 2.12 Ω . Critically damped is a decreasing line towards 0.

e) General total complex impedance (as impedance is other word for resistance) is equal to the sum of resistors in series as they are connected in series here. It is, therefore, equal to the sum of the three impedances: $Z = Z_R + Z_L + Z_C$. Which in turn leads to,

$$Z = R + i\omega L - \frac{i}{\omega C}$$

f) First determine the transfer A_L : $A_L = \frac{Z_L}{Z_R + Z_L + Z_C} = \frac{i\omega L}{R + i\omega L - \frac{i}{\omega C}}$

Multiply both numerator and denominator by $R - i(\omega L - \frac{1}{\omega C})$:

$$= \frac{i\omega L \cdot [R - i(\omega L - \frac{1}{\omega C})]}{R^2 + (\omega L - \frac{1}{\omega C})^2} = \frac{\omega L \cdot [(\omega L - \frac{1}{\omega C}) - iR]}{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

From which follows the voltage transfer, which is the modulus of A_L :

$$|A_L| = \frac{\sqrt{(\omega L)^2 \cdot [(\omega L - \frac{1}{\omega C})^2 + R^2]}}{\sqrt{[R^2 + (\omega L - \frac{1}{\omega C})^2]^2}} = \frac{\omega L}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

g) 15 kHz is equal to 94247.78 rad/s. Plug this in for ω at the formula of the voltage transfer, along with the previously found values to find $|A_L| = 0.02200489$.

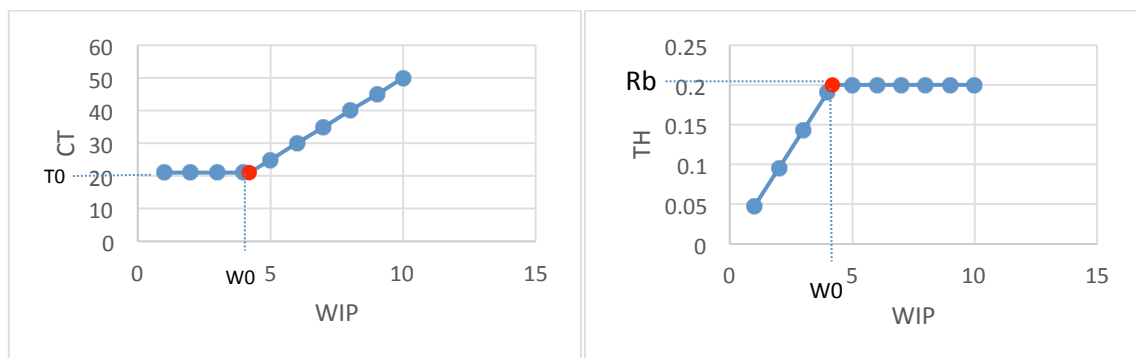
To translate this into decibels, use:

$$10 \cdot \log_{10}(0.02200489) = -16.574808 \text{ dB}$$

Using the inductance of 2.66 μ H, you will find an $|A_L|$ of 0.03431147 and -14.6456068 dB.

6) Business Dynamics

- a. WIP = work in progress [*products or jobs*]
the number of jobs/products that are being processed at a moment in time
TH = Throughput (rate) $\left[\frac{\text{products}}{\text{time}} \right]$
average output of a production process per unit of time.
CT = cycle time [*time*]
average time from release of a job at the beginning of the routing until it reaches the inventory at the end.
- b. Process 1, $(1/8)*2 = 0.25$ units/h.
Process 2, $1/5 = 0.2$ units/h.
Process 3, $1/2 = 0.5$ units/h.
Process 4, $(1/6)*2 = 0.33$ units/h.
Process 2 is the bottleneck, $r_b = 0.2$ units/h
- c. $T_0 = 8+5+2+6 = 21$ hours.
 $W_0 = 0.2*21 = 4.2$ units.
Graphs below



- d. The bullwhip effect refers to a trend of larger and larger swings in inventory and order size in response to changes in demand, as one looks at companies further back in supply chains.
Causes: Batching, demand processing/interpretation, demand forecasting, order batching, long delivery times, rationing gaming when there are shortages, price fluctuations, promotions/commercial activities.

e) $\gamma = (\beta + \epsilon)/2T$, $\omega_0 = \sqrt{1/T\tau} = 1/\sqrt{T\tau}$

f) Fill in γ and ω_0 found in e) in the mechanical solutions $\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$:

$\lambda_{1,2} = -(\beta + \epsilon)/2T \pm \sqrt{((\beta + \epsilon)/2T)^2 - (1/\sqrt{T\tau})^2} = -(1/2T)((\beta + \epsilon) \pm \sqrt{(\beta + \epsilon)^2 - (4T/\tau)})$. For dampening to occur, the first term $-(\beta + \epsilon)/2T$ should be negative. This means the boundary condition for the supply chain to be stable is that $(\beta + \epsilon)$ should be positive.