

Final Examination of Physical Systems

April 5, 2016, 9:00-12:00

INSTRUCTIONS (READ THIS CAREFULLY)

1. **Make sure you solve each problem on a separate sheet. Solutions of different problems appearing on the same sheet may be discarded.**
2. Write your name and student number clearly on top of every sheet.
3. Solutions which are not on the right sheet will not be graded. If you need extra paper, use the sheets provided and indicate for which problem the extra sheet is used for.
4. Solve the problems in a systematic way and check your answers. If you think you have made a mistake in a calculation, indicate this. Argue then how you intended to get the right answer and write this down.
5. This is a closed-book examination. No books, notes or graphical calculators may be used during the examination.
6. **Write in a legible manner. Unreadable text will not be handled during grading.**

Problem #1. Complex Numbers and Differential Equations (10 Points)

1. Consider the solution of a damped harmonic oscillation with the following expression for the displacement as a function of time:

$$x(t) = 9 \exp(-\gamma t) \exp(i\omega t)$$

Here, $\gamma = 0.2 \text{ s}^{-1}$ and $\omega = 4 \text{ rad/s}$.

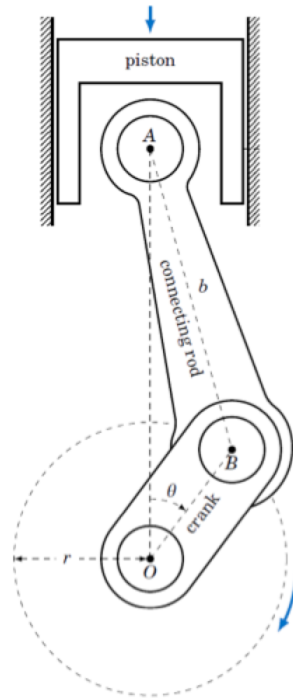
- a) Rewrite this solution in the form $x(t) = A(t) + iB(t)$. **(1 point)**
 - b) Calculate the displacements for $t = 0.0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0 \text{ s}$. **(3 points)**
 - c) Determine the first three t values for which the positive displacement is maximum after $t = 0$ and calculate the corresponding displacements. **(2 points)**
2. Consider the following differential equation:

$$5y'' + y' + 2.5y = 0$$

- a) Solve this differential equation given the following initial values $y(0) = 7$ and $y'(0) = 0$. **(3 points)**
- b) What is the limit of the solution as $t \rightarrow \infty$. **(1 point)**

Problem #2. Mechanical Oscillations without Damping (17 Points)

In the figure below, a schematic of a piston is shown from a car engine. The movement of the piston can, to a good approximation, be described by a harmonic oscillation. The crank rotates around point O with a speed of 2000 RPM (Rotations Per Minute) following a total path (circumference) of 0.3142 m. The weight of the piston is 0.5 kg. Assume there is no friction.



- Calculate the amplitude of the motion of the piston. **(3 points)**
- Let the crank start in its equilibrium point ($\theta = 0$) and give the solution to the equation of motion of the piston. **(3 points)**
- Calculate the maximum speed and acceleration of the piston. **(5 points)**
- Calculate the exerted force on the piston at the end of the stroke. **(2 points)**
- By doubling the speed of the engine (i.e. 4000 RPM), describe what will happen to the speed and the exerted force of the piston. **(4 points)**

Problem #3. Mechanical Oscillations with Damping (17 Points)

A mass $m = 1$ kg is horizontally connected to a spring with spring constant $k = 100$ N/m. The mass is sliding over the table with a friction coefficient c .

- a) Draw a picture of the setup and indicate all forces on the mass. Indicate the direction of motion as well. **(2 points)**
- b) Use the above picture to derive that the differential equation for the motion of the mass should be

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2x = 0$$

And specify γ and ω_0 in terms of m , k and c . **(2 points)**

- c) For which numerical value of c will the motion of the mass be critically damped? Also specify for which values of c the motion will be underdamped and overdamped. Give a sketch of x versus t for all three situations. **(4 points)**
- d) Assume from now on that the motion is underdamped. In this case, the solution of the differential equation will become

$$x(t) = B \cdot e^{-\gamma t} \cdot \cos(\omega t) \quad \text{assume } B > 0$$

specify ω in terms of c , m and k . Note that ω is different from ω_0 . **(1 point)**

- e) calculate the acceleration at $t = 0$. **(2 points)** For which numerical values of c will the acceleration be positive? **(1 point)** Also specify for which values it will be zero or negative. **(2 point)** Be sure to include the results of part c) into your answer.
- f) Give a sketch of x versus t for the 3 situations when the acceleration at $t = 0$ is positive, negative or zero. **(3 points)**

Problem #4. Electric Circuit – I (18 Points)

- a) Give the mathematical representation of both Kirchhoff laws and explain their physical meaning in one sentence (for each law). **(2 points)**

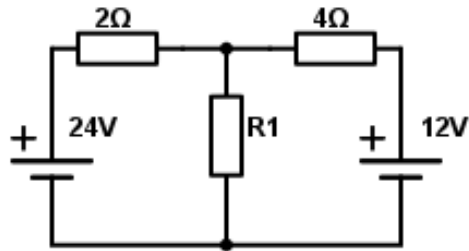


Figure 1: DC-circuit

- b) Consider the DC-circuit in Figure 1. Write down the KCL (Kirchhoff's Current Law) and KVL (Kirchhoff's Voltage Law) equations that are necessary to obtain the values for the currents through the individual components. **(4 points)**
- c) What value should the resistor R1 have in order to get a current of 3.82 A through the resistor with the value of 2 Ω? **(5 points)**

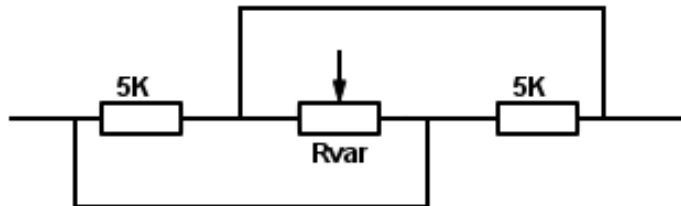


Figure 2: Variable resistor

- d) An electrician needs a variable resistor with a range between 1500 Ω and 2000 Ω. Unfortunately, he does not have such a resistor in stock. He decides to make the circuit shown in Figure 2. Write down an equation for the equivalent resistance of this schematic and work out it to obtain one fraction (ratio). **(4 points)** *Hint: Identify points with the same voltage when you measure over two points.*
- e) Calculate the range of the variable resistor he used in order to get the range he needed. **(3 points)**

Problem #5. Electric Circuit – II (18 Points)

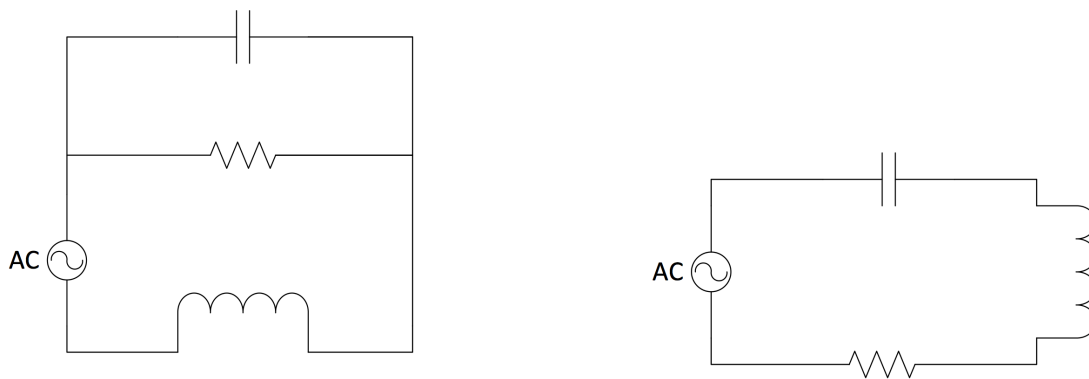


Figure 3: Left: Figure A). Right: Figure B)

Consider the circuit in the above figures.

- What is the complex impedance of the resistor? **(3 points)**
- What is the total impedance of circuit shown in figure A) **(4 points)** and the one in figure B)? **(2 points)**

Now consider Figure B) above, with $L = 300$ mH, $C = 70$ nF and $V_{in} = 12 \sin(\omega t)$

- Derive the transfer function for the capacitor. **(4 points)**
- In a circuit shown in Figure 3B, $\omega_{Res} = \frac{1}{2}\omega_0$, what is the R ? **(3 points)** *Hint:*

$$\omega_{RES} = \sqrt{\omega_o^2 - \frac{1}{2}\left(\frac{R}{L}\right)^2}.$$
- What is $|V_c|$ when $\omega = \omega_{Res}$ (use $R = 5000$ if you got no answer in D). **(2 points)**

Problem #6. Business Dynamics (20 Points)

Consider a production facility for Knørre book shelves. The production stations and their performance are given below.

Station number	Number of Machines	Processing Time (minutes)
1	2	15
2	1	5
3	4	20
4	8	20

- a) Calculate the critical work-in-progress W_0 . **(4 points)**
- b) Give a mathematical expression for the cycle-time in the best-case scenario, CT_{best} , for the above data and define its meaning. **(4 points)**
- c) Oscillations in a supply chain can be described by the following formula:

$$\frac{d^2q_i(t)}{dt^2} + \frac{\beta + \epsilon}{T} \frac{dq_i(t)}{dt} + \frac{1}{T\tau} q_i(t) = \frac{1}{T} \left(\frac{1}{\tau} q_{i+1}(t) + \beta \frac{dq_{i+1}(t)}{dt} \right) \quad (1)$$

which shows significant similarities with the formula for damped harmonic (mechanical) oscillations:

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t) \quad (2)$$

Explain the analogy between the right-hand sides of Eq. (1) and Eq. (2) **(2 points)**

- d) Name a well-known type of oscillation in Business Dynamics and describe its behaviour. **(4 points)**
- e) We can solve Eq. (1) and obtain the following values of λ for the solutions:

$$\lambda_{1,2} = -\frac{1}{2T}((\beta + \epsilon) \mp \sqrt{(\beta + \epsilon)^2 - \frac{4T}{\tau}})$$

Why do we require $\beta + \epsilon > 0$? **(2 points)**

- f) Which extra constraint do we add on the $\beta + \epsilon$ condition to ensure critical damping? **(3 points)** How does this constraint compare to the mechanical domain? **(1 point)**

**Solutions to the problems of the
Final Examination of Physical Systems
April 5, 2016**

Problem #1. Complex Numbers and Differential Equations (10 Points)

a) $9 \exp(-0.2t) \cos(4t) + 9i \exp(-0.2t) \sin(4t)$

b)

t	0.0	0.5	1.0	1.5	2.0	2.5	3.0
x	9	-3.39	-4.82	6.40	-0.88	-4.58	4.17

c) Max at $t = 1.57, 3.14, 4.71$. These correspond to $x = 6.57, 4.80, 3.51$.

d) $y = 7 \exp(-0.1t) \cos(0.7t) + \exp(-0.1t) \sin(0.7t)$

e) 0

Problem #2. Mechanical Oscillations without Damping (17 Points)

- a) Amplitude is the same as length $OB = r = \frac{0.3142}{(2\pi)} = 0.05$ m
- b) At point ($\theta = 0$), the piston is in its maximum position, so the equation of motion is given by: $x(t) = A \cos(\omega t)$.
- c) $\omega = 2\pi f = 2\pi \frac{2000}{60} = 209.45$ rad/s

$$x(t) = A \cos(\omega t)$$

$$v = x'(t) = -\omega A \sin(\omega t) \Rightarrow v_{max} = \omega A = 209.45 \cdot 0.05 = 10.47 \text{ m/s}$$

$$a = x''(t) = \omega^2 A \cos(\omega t) \Rightarrow a_{max} = \omega^2 A = (209.45)^2 \cdot 0.05 = 2193.25 \text{ m/s}^2$$

- d) $F = ma = 0.5 \cdot 2193.25 = 1096.6$ N
- e) By doubling the RPM of the engine, the new frequency will be 2ω . So the new speed will increase by a factor of 2, $v_{(max,new)} = 2\omega A$ and the force will increase by a factor of 4 since $a_{(max,new)} = (2\omega)^2 A = 4\omega^2 A$.

Problem #3. Mechanical Oscillations with Damping (17 Points)

a) See the figure.

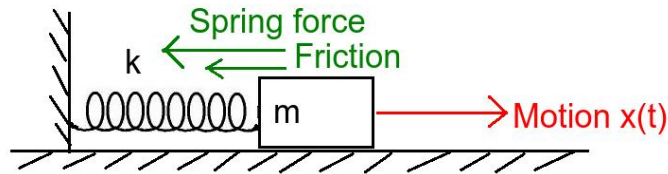


Figure 1: Setup for this question.

b) Newtons second law gives that

$$\Sigma F = F_{spring} + F_{Friction} = ma$$

$$-kx - cv = ma$$

$$ma + cv + kx = 0$$

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{c}{m} \frac{dx}{dt} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = 0 \quad \gamma = \frac{c}{2m} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

c) A critically damped motion means that $\gamma = \omega_0$. Filling this in gives

$$\frac{c}{2m} = \sqrt{\frac{k}{m}}$$

$$c = 2m \cdot \sqrt{\frac{k}{m}} = \sqrt{4km} =$$

$$\sqrt{4 \cdot 1 \text{ kg} \cdot 100 \text{ N/m}} = 20 \text{ Ns/m}$$

Hence we have critical damping at $c = 20 \text{ Ns/m}$, we have overdamping at $c > 20 \text{ Ns/m}$ and underdamping at $c < 20 \text{ Ns/m}$.

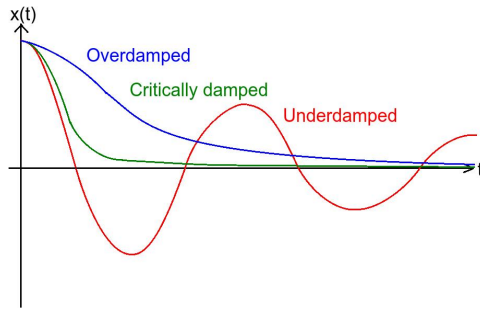


Figure 2: Sketch of different dampings.

d)

$$\omega^2 = \omega_0^2 - \gamma^2 =$$

$$\frac{k}{m} - \left(\frac{c}{2m}\right)^2 = \frac{k}{m} - \frac{c^2}{4m^2} = \frac{4mk - c^2}{4m^2}$$

$$\omega = \sqrt{\frac{4mk - c^2}{4m^2}}$$

e) Take the solution

$$x(t) = B \cdot e^{-\gamma t} \cdot \cos(\omega t)$$

And differentiate this to time:

$$x'(t) = -\gamma B \cdot e^{-\gamma t} \cdot \cos(\omega t) - \omega B \cdot e^{-\gamma t} \cdot \sin(\omega t)$$

Then differentiate it again:

$$x''(t) = +\gamma^2 B \cdot e^{-\gamma t} \cdot \cos(\omega t) + \gamma \omega B \cdot e^{-\gamma t} \cdot \sin(\omega t) + \omega \gamma B \cdot e^{-\gamma t} \cdot \sin(\omega t) - \omega^2 B \cdot e^{-\gamma t} \cdot \cos(\omega t) =$$

$$(\gamma^2 - \omega^2) \cdot B e^{-\gamma t} \cdot \cos(\omega t) + 2\gamma \omega B e^{-\gamma t} \cdot \sin(\omega t)$$

Now fill in that $t = 0$:

$$x''(t = 0) = (\gamma^2 - \omega^2) \cdot B \cdot 1 \cdot \cos(0) + 2\gamma \omega B \cdot 1 \cdot \sin(0) = (\gamma^2 - \omega^2) \cdot B$$

Now since $B > 0$, the acceleration will be positive if

$$\gamma^2 - \omega^2 > 0$$

Hence

$$\gamma^2 > \omega^2$$

Giving

$$\gamma > \omega$$

Filling in what we obtained for ω and γ . this becomes:

$$\frac{c}{2m} > \sqrt{\frac{4mk - c^2}{4m^2}}$$

Squaring gives:

$$\frac{c^2}{4m^2} > \frac{4mk - c^2}{4m^2} = \frac{k}{m} - \frac{c^2}{4m^2}$$

Hence:

$$\frac{c^2}{2m^2} > \frac{k}{m}$$

Hence:

$$\frac{c^2}{2m} > k$$

Hence:

$$c^2 > 2km$$

Resulting in:

$$c > \sqrt{2km} = \sqrt{2 \cdot 1 \text{ kg} \cdot 100 \text{ N/m}} = 14.14 \text{ Ns/m}$$

By the same reasoning the acceleration will be zero when $c = 14.14 \text{ Ns/m}$ and negative when $c < 14.14 \text{ Ns/m}$. Now to include question 3, one should realize that the values just obtained for c are based on the underdamped solution. Hence they are only valid in the case of underdamping, or when $c < 20 \text{ Ns/m}$. Hence our final answer is:

- We get negative acceleration at $t = 0$ for $c < 14.14 \text{ Ns/m}$.
- We get zero acceleration at $t = 0$ for $c = 14.14 \text{ Ns/m}$.
- We get positive acceleration at $t = 0$ for $14.14 \text{ Ns/m} < c < 20 \text{ Ns/m}$.

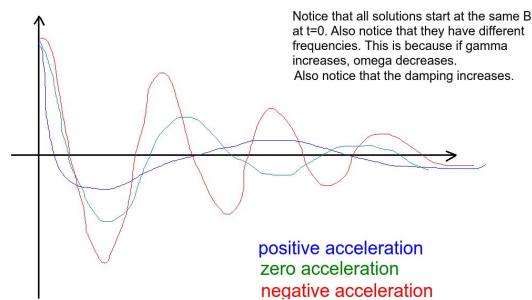


Figure 3: Sketch of different accelerations.

f)

Problem #4. Electric Circuit – I (18 Points)

a)

$$\Sigma I = 0 \text{ at knots}$$

$$\Sigma V = 0 \text{ in loops}$$

b)

$$-24 + 2I_1 + R_1 \cdot I_3 = 0 \quad (1)$$

$$-R_1 \cdot I_3 + 4I_2 + 12 = 0 \quad (2)$$

$$I_1 - I_2 - I_3 = 0 \quad (3)$$

c) from 1:

$$-16.36 + R_1 \cdot I_3 = 0 \quad (4)$$

from 2 and 3:

$$-I_3(R_1 + 4) + 27.28 = 0 \quad (5)$$

from 4 and 5:

$$I_3 = \frac{27.28}{R_1 + 4} = \frac{16.36}{R_1}$$

$$R_1 = 0.6(R_1 + 4)$$

$$R_1 = 6 \Omega$$

d) 3 resistors in parallel:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_{var}} + \frac{1}{R_2}$$

$$R_{eq} = \frac{R_1 R_{var} R_2}{R_{var} R_2 + R_1 R_2 + R_1 R_{var}}$$

e)

$$R_{eq} = \frac{2.5 \cdot 10^7 R_{var}}{10^4 R_{var} + 2.5 \cdot 10^7}$$

$$R_{var} = \frac{2.5 \cdot 10^7 R_{eq}}{2.5 \cdot 10^7 - 10^4 R_{eq}}$$

thus the range of the variable resistor should be between 3750Ω and 10000Ω .

Problem #5. Electric Circuit – II (18 Points)

a) R

b) Figure 3A: $Z_{tot} = \frac{1}{\frac{1}{R} + R(\omega C)^2} - i\left(\frac{\omega C}{\frac{1}{R^2} + (\omega C)^2} - \omega L\right)$

Figure 3B: $Z_{tot} = R + i\omega L - \frac{i}{\omega C}$

c) Formulas V.47 and V.48 of the syllabus 2016.

d) Syllabus 2016, formula V.65 ($\omega_{RES} = \sqrt{\omega_o^2 - \frac{1}{2}\left(\frac{R}{L}\right)^2}$) and V.66 ($\omega_0 = \frac{1}{\sqrt{LC}}$) \rightarrow
 $R = \sqrt{\frac{3}{2}} \cdot \frac{L}{C} = 2535.5 \Omega$.

e) $|A_c| = 1.039$ giving $|V_c| = 12.46|\sin(\omega t)|$ or $8.44|\sin(\omega t)|$ (if one uses $R = 5000 \Omega$).

Problem #6. Business Dynamics (20 Points)

a) Critical WIP lies at $W_0 = r_b \times T_0$.

SN	NoM	PT (minutes)	Production Rate (units/h)
1	2	15	8
2	1	5	20
3	4	20	12
4	8	20	24

$r_b = 8$ units/hour. $T_0 = 15 + 5 + 20 + 20 = 1$ hour.

$$W_0 = 8 \text{ units}$$

- b) See equation VI.1 from reader (version 2016).
- c) The right-hand sides of both equations represent a driving force/effort. For mechanical oscillations this is an actual force, for BD-oscillations this is the next-stage demand.
- d) The bullwhip effect. It refers to a trend of larger and larger swings in inventory in response to changes in demand, as one looks at companies further back in the supply chain of a product. The following words or synonyms should be used correctly to obtain full points: swings (oscillations), supply chain, demand changes.
- e) This enforces dampening to occur. If $\beta + \epsilon < 0$ the oscillations would be growing in amplitude and the system would be unstable (i.e. it would blow up).
- f) For critical dampening we require

$$\beta + \epsilon = 2\sqrt{\frac{T}{\tau}}$$

This is equivalent to $\gamma^2 = \omega_0^2$ in the mechanical domain.