

# Re-Examination of Physical Systems

May 1, 2014, 18:30-21:30

## **INSTRUCTIONS (READ THIS CAREFULLY)**

1. **Make sure you solve each problem on a separate sheet.**
2. Write your name and student number clearly on top of every sheet.
3. Solutions which are not on the right sheet will not be graded. If you need extra paper, use the sheets provided and indicate for which problem the extra sheet is used for.
4. Solve the problems in a systematic way and check your answers. If you think you have made a mistake in a calculation, indicate this. Argue then how you intended to get the right answer and write this down.
5. This is a closed-book examination. No books, notes or graphical calculators may be used during the examination.
6. There are 6 problems in this exam each with its own parts.
7. **Write in a legible manner. Unreadable text will not be handled during grading.**

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**Problem #1. Complex Numbers and Differential Equations (10 Points)**

Consider the complex function:

$$x(t) = 2e^{-2t}e^{i\omega t}$$

- a) Make a sketch of the function  $x(t)$  in the complex plane. Indicate the direction of increasing  $t$ . (Note: a sketch of the general form is sufficient, don't worry about the precise values for  $x(t)$ .) **(2 Points)**
- b) Write  $x(t)$  as  $x(t) = A(t) + iB(t)$ . **(2 Points)**
- c) What is the norm  $|x(t)|$ ? At which value for  $t$  has the norm decreased to half its value at  $t = 0$ ? **(3 Points)**

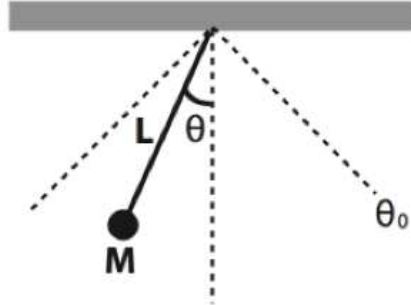
Now consider the differential equation:

$$\frac{dx(t)}{dt} = \frac{t^2}{x(t)}$$

- d) Find the solution  $x(t)$  to the above differential equation. **(3 Points)**

**Problem #2. Mechanical Oscillations without Damping (17 Points)**

Consider a mathematical pendulum like the one in the figure.



The pendulum consists of a massless, rigid bar with a length  $L$ , and with a mass  $m$  hanging at the end. We assume that there is no friction acting on the motion of the pendulum (no damping).

- a) Give the expression of the two components of the gravity force acting on the mass (longitudinal and tangential) as a function of  $\theta$ . **(4 Points)**
- b) The motion of the pendulum can be treated as a harmonic motion, described by the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

where  $g$  is the acceleration due to gravity ( $9.81 \text{ m/s}^2$ ). Under which assumption is this equation really valid to describe the motion of the pendulum? Give the expression for the angular frequency  $\omega$  of the motion. **(3 Points)**

- c) Show that  $\theta(t) = \theta_0 \sin(\omega t + \alpha)$  is a solution of the equation of motion. **(3 Points)**

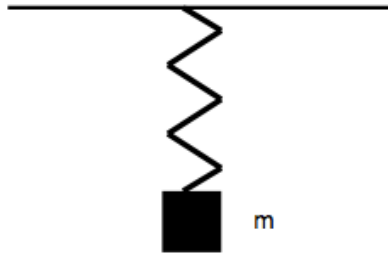
Consider now  $L = 15 \text{ m}$  and  $\theta_0 = 15^\circ$ .

- d) Calculate the maximum speed of the mass during the oscillation. Hint: use the conservation of energy and the geometrical relation between  $\theta$  and the height  $h$ . **(3 Points)**
- e) What is the value of  $\theta$  when the speed is  $2 \text{ m/s}$ ? **(4 Points)**

**Problem #3. Mechanical Oscillations with Damping (17 Points)**

A damped oscillation of a spring with mass ( $m$ ) connected to one side (see figure) is described by the following differential equation:

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2x = 0$$

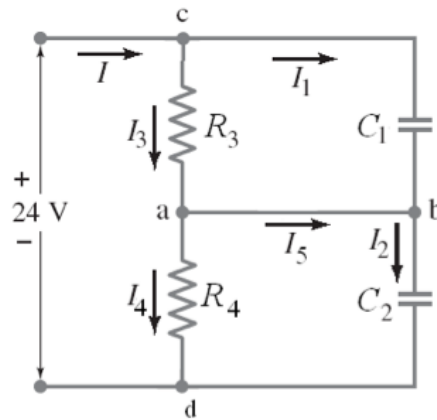


- a) Use the forces active in this damped oscillation to show that the above differential equation is correct. (HINT:  $F_{friction} = -2\gamma m \frac{dx}{dt}$ ) **(4 Points)**
- b) What happens when  $\gamma^2 = \omega_0^2$  and when  $\gamma^2 > \omega_0^2$ ? You can also use a qualitative sketch of the situations. **(3 Points)**

$x(t) = Ae^{-\gamma t} \cos\left(\sqrt{\omega_0^2 - \gamma^2}t\right)$  is a solution to the above differential equation.

- c) Calculate the time at which the amplitude is less than 5% of its original value if  $\gamma = 0.1 \text{ s}^{-1}$ . **(5 Points)**
- d) What is the speed at  $t = 0 \text{ s}$  if  $A = 0.1 \text{ m}$ ? **(5 Points)**

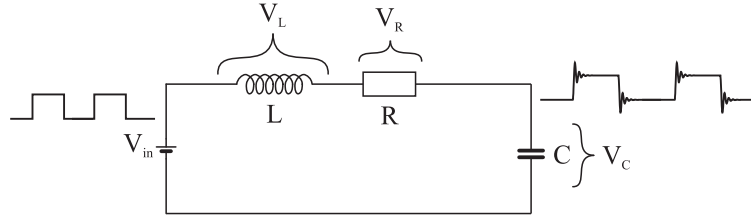
**Problem #4. Electric Circuit – DC (18 Points)**



- Describe Kirchhoff's 1<sup>st</sup> and 2<sup>nd</sup> laws. **(6 Points)**
- Write down the Kirchhoffs 1<sup>st</sup> law (current) for junctions c & d. Write down Kirchhoffs 2<sup>nd</sup> law (voltage) for each of the three loops. **(5 Points)**
- Determine the expression for the time constant or characteristic time (in terms of  $C_1$ ,  $C_2$ ,  $R_3$ ,  $R_4$ ) for the entire circuit. (Hint: When solving the equations, try to bring the form close to what you are familiar with without actually solving the differential equation from which the time constant should be extracted. You can also solve the appropriate differential equation and obtain the characteristic time. Also, bear in mind while solving equations that current is the rate of change of charge.) **(5 Points)**
- Given that  $C_1 = 0.48 \mu\text{F}$ ,  $C_2 = 0.36 \mu\text{F}$ ,  $R_3 = 4.4 \Omega$ , and  $R_4 = 8.8 \Omega$ , calculate the time constant or characteristic time in  $\mu\text{s}$  (microseconds). **(2 Points)**

### Problem #5. Electric Circuit – AC (18 Points)

Consider the circuit shown in the figure:



For which  $L = 1.0 \text{ mH}$ ,  $C = 10.0 \mu\text{F}$ ,  $R = 0.1 \Omega$  and  $V_{in}$  can step between 0 and 2 V.

- a) Use the second law of Kirchhoff (related to voltages in a loop) to derive the following differential equation for the circuit:

$$\frac{d^2 I}{dt^2} + \frac{R}{L} \frac{dI}{dt} + \frac{1}{LC} I = 0$$

Show clearly each step in your derivation. HINT: Apply Kirchhoff's law and take the derivative of the result. **(4 Points)**

- b) A solution function of the differential equation in part a) is

$$I(t) = I_0 e^{-\frac{R}{2L}t} \sin(\omega t)$$

which describes the response of the system after each switch in input voltage. The amplitude of the steps at  $V_{in}$  is 2 V. Calculate  $I_0$ . Hints: At time  $t = 0$ , all of the applied voltage will be across the inductor. Frequency  $\omega$  is the frequency of an undriven free damped oscillator. Approximate  $\omega$  with  $\omega_0$ . **(4 Points)**

- c) Note that one complete cycle of the input signal contains two steps: once up, and once down. It is given that the oscillations dampen to 1% of the starting amplitude  $I_0$  just before the next step occurs. Calculate the frequency of the square wave *input*. **(4 Points)**
- d) How would you change the component values if you needed to double the oscillation frequency  $\omega$ , but keeping the capacitance and the damping time constant (characteristic time) the same? For simplicity, you may substitute  $\omega_0$  for  $\omega$ . **(3 Points)**
- e) Explain: what would happen if we were to change the component values such that  $\frac{1}{2} \frac{R}{L} = \omega_0$ ? (Hint: compare the coefficients in the equation above with the mechanical situation.) What is this condition called? **(3 Points)**

### Problem #6. Business Dynamics (20 Points)

- a) In business dynamics it is important to work from exact and correct definitions. Give the definition of the raw processing time  $T_0$ . Does the raw processing time include waiting times, e.g. for a machine to be available for the next production step? Give the definition of the critical Work-In-Progress in words. **(2 Points)**

Consider a production line of 8 stations all having a single machine. The performance of the line has been measured during steady state operation, yielding the following results:

Station	Average throughput rate [#units/hour]	Average processing time [hour]
1	115	4.7
2	113	5.4
3	102	0.6
4	198	8.1
5	200	3.5
6	251	6.2
7	136	7.8
8	121	4.7

The analysis further showed that the average throughput TH was 360 units per day, on average there were 8000 units on the production floor (WIP) and the average cycle time (CT) for a unit equalled 23 days. Production is done in 8 hours per day (1 shift).

- b) Determine  $r_b$  and  $T_0$  for this production line and subsequently calculate the critical WIP level,  $W_0$ . Which law did you use to calculate  $W_0$ ? Give both the formulae and the name of the special case of the law you used. **(3 Points)**
- c) Give the formulae for the throughput TH in the Practical Worst Case (PWC). Now compare the actual throughput of the production line with  $TH_{PWC}$ . What can you conclude from this? (If no reliable values for  $r_b$ ,  $T_0$ , and  $W_0$  were obtained under b), you may use 80 units/hour, 30 hour and 2400 units). **(3 Points)**
- d) Now repeat the study graphically in terms of Cycle Time. Draw the Cycle Time as a function of Work-In-Progress. Draw in the Best-case, Worst-case and the Practical-Worst-case of this production line. Clearly indicate the numerical values along the axes. Indicate the good and the bad region and mark the actual performance of the production line. Again draw your conclusion. **(3 Points)**

- e) The analyses under c) and d) should lead to similar conclusions. You have to choose one of the two to present your findings to management. Which one would you choose and why? **(1 Point)**

The article of Lee et al. describes the effect of shortage gaming on supply chains. Usually, supply chains suffer from oscillations due to the lack of knowledge on order behavior in the supply chain. Even when regular order behavior is predictable, oscillations can occur when a supplier runs out of stock.

- f) Describe what effect causes oscillations when running out of stock **according to Lee et al.**, and how does this effect work? Give an example. **(2 Points)**
- g) Lee et al. also describe three solutions to reduce supply chain oscillations, what can be done to reduce the randomness of orders from customers? Give two examples **from the article**. **(2 Points)**

When talking about oscillations, the reader describes a differential equation from which these oscillations in a supply chain can be explained:

$$\frac{d^2 q_i(t)}{dt^2} + \frac{(\beta + \epsilon)}{T} \frac{dq_i(t)}{dt} + \frac{1}{T \cdot \tau} q_i(t) = \frac{1}{T} \left( \frac{q_{i+1}(t)}{\tau} + \beta \frac{dq_{i+1}(t)}{dt} \right)$$

Just as in the electrical and in the mechanical domain, damping can occur to reduce oscillations.

- h) Determine the damping,  $\gamma$ , and the eigen frequency,  $\omega_0$ , for this supply chain. **(2 Points)**
- i) An electrical systems is driven by a power supply, a mechanical system is driven by an external force. A supply chain is driven by  $\frac{1}{T} \left( \frac{q_{i+1}(t)}{\tau} + \beta \frac{dq_{i+1}(t)}{dt} \right)$ . Explain what exactly  $q_{i+1}(t)$  is. Why does it make sense from a business perspective that this is the driving force of a supply chain? **(2 Points)**

# ANSWERS TO QUESTIONS OF THE RE-EXAMINATION OF PHYSICAL SYSTEMS, MAY 1, 2014

## PROBLEM 1

- a) Draw a sketch of a spiral, starting at the  $x$ -axis, and spiraling inwards counter-clockwise.
- b) Use Euler's formula to write

$$x(t) = 2e^{-2t} (\cos(\omega t) + i \sin(\omega t))$$

Then explicitly:

$$A(t) = 2e^{-2t} \cos(\omega t) \quad \text{and} \quad B(t) = 2e^{-2t} \sin(\omega t).$$

- c) The norm of  $x(t)$  is simply

$$|x(t)| = 2e^{-2t}.$$

At  $t = 0$ , this is  $|x(0)| = 2$ , so we want to find  $t_*$  for which  $|x(t_*)| = 1$ . A simple calculation shows

$$e^{-2t_*} = \frac{1}{2} \quad \text{so:} \quad -2t_* = \ln\left(\frac{1}{2}\right), \quad \rightarrow \quad t_* = -\frac{1}{2} \ln\left(\frac{1}{2}\right) = 0.3466 \text{ s}$$

- d) Derivation:

First write:  $x(t)dx = t^2 dt$  and integrate

$$\text{Find that: } \frac{1}{2}x^2(t) = \frac{t^3}{3} + c$$

$$\text{So finally: } x(t) = \pm \sqrt{\frac{2t^3}{3} + c'}$$

## Problem 2

- a) Gravity components:

- Longitudinal:  $F_L = mg \cos(\theta)$
- Tangential:  $F_T = mg \sin(\theta)$

- b) The equation is valid if the angle of oscillation  $\theta$  is small. The angular frequency is

$$\omega = \sqrt{\frac{g}{L}}$$

- c) To show that  $\theta(t) = A \sin(\omega t + \alpha)$  is a solution they have to take it, and its second derivative, and put them into the equation to obtain  $0 = 0$ .

- d) Conservation of energy:  $E_{tot} = E_{kin} + E_{pot}$

Highest point of the motion: the kinetic energy is 0, the potential energy has its maximum value. The vertical distance from the level of the lowest point is:

$$h = L - L \cos \theta_0$$

$$E_{pot, max} = mgL(1 - \cos \theta_0)$$

At the lowest point of the motion, the potential energy is 0 and the kinetic energy has its maximum value, because in that point the body has its maximum speed:

$$E_{kin, max} = \frac{1}{2} m v_{max}^2 = mgL(1 - \cos \theta_0)$$

$$v_{max} = \sqrt{2gL(1 - \cos \theta_0)} = 3.17 \text{ m/s}$$

- e) When the speed is 2 m/s, the body has both kinetic and potential energy.

However, their sum is constant and therefore

$$E_{kin} + E_{pot} = E_{pot, max} = E_{kin, max}$$

At this moment, the body is at a height  $h_1$  from the lowest level. The equation expressing the conservation of energy reads

$$\frac{1}{2} m v_1^2 + m g h_1 = \frac{1}{2} m v_{max}^2$$

From which,  $h_1 = \frac{1}{2} (v_{max}^2 - v_1^2) \frac{1}{g} = 0.31 \text{ m}$  giving an angle of  $11.5^\circ$ .

### Problem 3

- a) Write all the force terms:  $F_{tot} = m \cdot \frac{d^2 x}{dt^2}$ ,  $F_{fr.} = -2\gamma m \frac{dx}{dt}$ , and  $F_{spring} = -kx$  to get:

$$\frac{d^2 x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = 0$$

- b)  $\gamma^2 = \omega_0^2$  Critical damping. The system goes back to equilibrium fastest.

$\gamma^2 > \omega_0^2$  Over-damped. The system goes back to equilibrium slower than critical damping.

- c)  $e^{-\gamma t} = 0.05$  gives  $t=30 \text{ s}$ . So, for  $t > 30 \text{ s}$

d)

$$v = \frac{dx}{dt} = Ae^{-\gamma t} \cdot (-\gamma) \cdot \cos\left(\sqrt{\omega_0^2 - \gamma^2} t\right) + Ae^{-\gamma t} \cdot \left(-\sqrt{\omega_0^2 - \gamma^2}\right) \cdot \sin\left(\sqrt{\omega_0^2 - \gamma^2} t\right)$$

So, at  $t=0$ ,  $v=-A\gamma=-0.01$  m/s.

#### Problem 4

a) As in the lecture notes.

$$b) \quad I = I_1 + I_3 \quad I = I_2 + I_4$$

$$V - \frac{Q_1}{C_1} - \frac{Q_2}{C_2} = 0 \quad \frac{Q_1}{C_1} - I_3 R_3 = 0 \quad \frac{Q_2}{C_2} - I_4 R_4 = 0$$

c) Use the above equations, differentiate the first one and play with all the elements to get the form that you are familiar with for one resistor and one capacitor in series with the power supply. From here, you can obtain the RC time:

$$\tau = RC = R_4 \left( \frac{C_1 + C_2}{C_1} \right) C_1 \left( \frac{R_3}{R_4 + R_3} \right) = \frac{R_3 R_4 (C_1 + C_2)}{R_3 + R_4}$$

$$d) \quad \tau = 2.5 \mu s$$

#### Problem 5

$$a) \quad V_C = \frac{Q}{C} \quad V_L = L \frac{dI}{dt} \quad V_R = IR$$

Kirchoff 2  $\rightarrow$

$$V_C + V_L + V_R = 0 \rightarrow \frac{Q(t)}{C} + L \frac{dI(t)}{dt} + I(t)R = 0$$

Use the fact that the charging rate of the capacitor is equal to the current, and take derivatives:

$$\frac{I(t)}{C} + L \frac{d^2 I(t)}{dt^2} + \frac{dI(t)}{dt} R = 0$$

Divide by L:

$$\frac{I(t)}{LC} + \frac{d^2 I(t)}{dt^2} + \frac{dI(t)}{dt} \frac{R}{L} = 0$$

b) The current is:

$$I(t) = I_0 e^{-\frac{1}{2L}t} \sin(\omega t)$$

At  $t=0$  we have an instantaneous change in input voltage. If there were no coil, the charging current would simply be  $V_{in}/R$ . But since there is a coil, at least for an infinitely small moment at  $t=0$ , all of the voltage of  $V_{in}$  will be across the coil.

That voltage is:

$$\begin{aligned} V_L = V_{in} &= L \frac{dI}{dt} = L \frac{d}{dt} \left( I_0 e^{-\frac{1}{2L}t} \sin(\omega t) \right)_{t=0} \\ &= L * I_0 \left( -\frac{1}{2L} e^{-\frac{1}{2L}t} \sin(\omega t) + \omega e^{-\frac{1}{2L}t} \cos(\omega t) \right)_{t=0} \end{aligned}$$

At  $t=0$ :

$$V_{in} = L * I_0 \omega$$

Approximating  $\omega \approx \omega_0$

$$I_0 = \frac{V_{in}}{L\omega} = \frac{V_{in}\sqrt{LC}}{L} = V_{in} \sqrt{\frac{C}{L}}$$

$$\text{Check units: } \text{volt} \sqrt{\frac{\text{coulomb/volt}}{(\text{volt s}^2/\text{coulomb})}} = \text{volt} \sqrt{\frac{\text{coulomb}^2}{(\text{volt}^2 \text{s}^2)}} = \frac{\text{coulomb}}{\text{s}}.$$

So hey, this might actually work.

So we have:

$$I_0 = V_{in} \sqrt{\frac{C}{L}} = 2 \sqrt{\frac{10 * 10^{-6}}{1 * 10^{-3}}} = 2\sqrt{1 * 10^{-2}} = 0.2 \text{ A}$$

c) First calculate the time at which the oscillation amplitude drops below 1%:

$$e^{-\frac{1}{2L}t} = 0.01 \rightarrow -\frac{1}{2L}t = \ln(0.01)$$

$$t = -\ln(0.01) * 2 \frac{L}{R}$$

That time corresponds to the time between two changes of the input voltage. So, the actual time duration of one cycle of the input square wave is twice that:

$$T_{in} = -\ln(0.01) * 4 \frac{L}{R}$$

frequency:

$$\begin{aligned} f &= \frac{1}{T_{in}} = \frac{-1}{\ln(0.01) * 4 \frac{L}{R}} = \frac{-R}{\ln(0.01) * 4L} = \frac{-0.1}{\ln(0.01) * 4 * 1 * 10^{-3}} \\ &= \frac{-100}{4 * \ln(0.01)} = \frac{-100}{4 * (-4.61)} = 5.4 \text{ Hz} \end{aligned}$$

- d) Doubling  $\omega$ , but keeping the capacitance and damping time constant the same

$$\text{Approximating } \omega \approx \omega_0 = \frac{1}{\sqrt{LC}}$$

So that means  $L_{\text{new}} = 1/4 * L_{\text{old}} = 1/4 * 1 \text{ mH} = 0.25 \text{ mH}$

However  $R/L$  should remain the same  $\rightarrow R = R/4 = 0.025 \Omega$

- e) In the given equation for the current in section (b) one can recognize the damping constant  $\gamma = \frac{1}{2} \frac{R}{L}$

The condition is that of  $\gamma = \omega_0$ , which is the special case which is called critical damping. In that case we no longer have an oscillating solution – only a modified exponential decay. It is usually called ‘the fastest’ way to return to equilibrium *without overshooting the equilibrium point*. If we decrease  $\gamma$  to a value below the condition above, we overshoot. If we increase, the return to equilibrium will not overshoot, but becomes slower.

## Problem 6

- a)  $T_0 = \Sigma$  cycle time of the stations on the process line

No

Critical Work-in-Progress: the WIP that maximizes the throughput while minimizing the cycle time. Or  $W_0 = r_b \times T_0$

- b)  $r_b = 102 \text{ units/h}$

$$T_0 = 41 \text{ h}$$

$$W_0 = 102 \times 41 = 4182 \text{ units}$$

$W_0 = r_b \times T_0$ ; special case of Little’s Law.

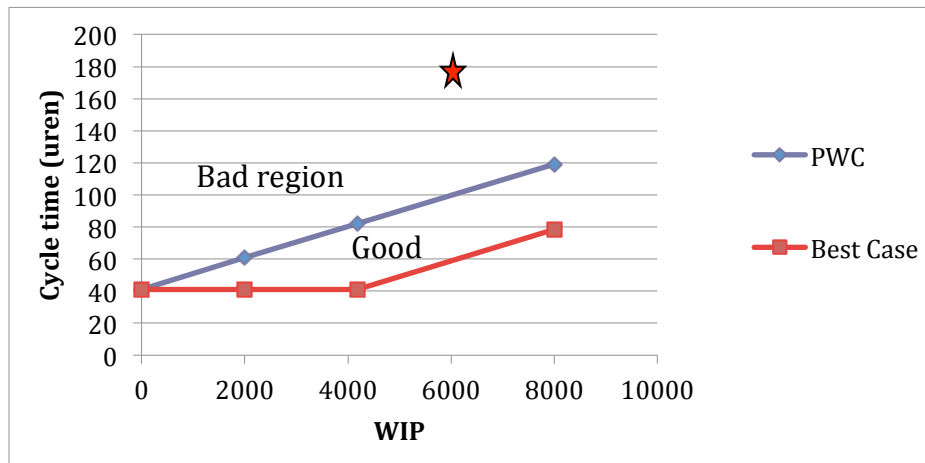
- c)  $TH_{PWC} = \frac{WIP}{CT_{PWC}} = \frac{w}{W_0 + w - 1} \cdot r_b$

$$TH_{\text{actual}} = 360/8 = 45 \text{ units/day}$$

$$TH_{\text{pwc}} = 8000 / (4182 + 8000 - 1) \times 102 = 67 \text{ units/day}$$

$$(TH_{\text{pwc}} = 8000 / (2400 + 8000 - 1) \times 80 = 62 \text{ units/day}).$$

Conclusion: the actual performance is 1.5 that of the practical worst case.



The worst case is almost equal to the y-axis.

The actual case WIP=6000 and CT = 23 days =  $23 \times 8 = 184$  h.

- d) The throughput could be higher, but CT really shows how dramatic the situation is. A given unit takes 23 days = 184 hours to complete a single unit, 4.5 times the raw processing time. The WIP is high, so this immediately also indicates there waste. A graphical representation is practical for management and it indicates what can be reached, what can be reached and that is can be even worse.
- e) Both analyses yield a similar result. A graphic representation is often preferred by management, as it gives a direct visual view on the magnitude of the problem. (Other reasoning with a good underpinning is accepted.)
- f) This is called shortage gaming. When suppliers run out of stock, customers will start looking for additional suppliers, whoever delivers first, gets paid, all other orders are cancelled. This will cause temporary fluctuations in ordering behavior, not knowing when a competitor will deliver. This causes oscillations.
- g)
  1. Provide discounts in order cost when accepting a set order interval.
  2. Set a minimum order size (batch size).
  3. Reduce lead times of production to anticipate more quickly on orders.

h)

$$\gamma = \frac{\beta + \varepsilon}{2T}$$
$$\omega_0 = \sqrt{\frac{1}{T\tau}}$$

- i) ( $Q_{i+1}(t)$  is the delivery rate to the customer as a function of time.)  
 $q_{i+1}(t)$  is the deviation from the average delivery rate to the customer as a function of time.  
A supply chain is customer driven.